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Assouad dimensions of Moran sets

Jinjun Li

Department of Mathematics, Zhangzhou Normal University, Zhangzhou, 363000, PR China

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ABSTRACT

We prove that the Assouad dimensions of a class of Moran sets coincide with their upper box dimensions and packing dimensions.

RÉSUMÉ

Nous montrons que, pour les ensembles d'une classe de Moran, la dimension d'Assouad coïncide avec la dimension de boîte supérieure et avec la dimension d'empilement.

1. Introduction

Let us begin with the definition of the Assouad dimension which is introduced by Assouad [1]. A metric space X is doubling if there exists an $N > 0$ such that any ball can be covered by N balls of half the radius. Repeated applying this property, we see that there exist some $b, c > 0$ and $\alpha > 0$ such that for any r, R satisfying $0 < r < R < b$, any ball $B(x, R)$ can be covered by $c(\frac{R}{r})^\alpha$ balls of radius r. The Assouad dimension of a metric space X, denoted by $\dim_A X$, is the infimal value of α for which there exists a constant c such that the above property holds. More precisely, for $r, R > 0$, let $N_{r,R}(X)$ denote the smallest number of balls with radii equal to r needed to cover any ball with radius equal to R, then

$$\dim_A X = \inf\Bigg\{ \alpha \geqslant 0 \,\Bigg|\, \text{there are constants } b, c > 0 \text{ satisfying:} \\ \text{for any } 0 < r < R < b,\ \text{the inequality } N_{r,R}(X) \leqslant c\left(\frac{R}{r}\right)^\alpha \text{ holds}\Bigg\}. \tag{1.1}$$

The Assouad dimension plays an important role in the study of quasi-conformal mappings in $\mathbb{R}^d$, see [3,6]. However, it has received little attention on fractal geometry. It is well known that

$$\dim_H X \leqslant \dim_P X \leqslant \overline{\dim}_B X \leqslant \dim_A X, \tag{1.2}$$

where $\dim_H X, \dim_P X, \overline{\dim}_B X$ denote the Hausdorff, packing and upper box dimensions of X, respectively. We refer the reader to [2,9] for the definitions and basic properties of these fractal dimensions. It is worth to point out that the last inequality in (1.2) may be strict. For example, let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$, then $\underline{\dim}_B X = \overline{\dim}_B X = \frac{1}{2}$, but $\dim_A X = 1$, see Example 3.5 in [2] and Exercise 10.16 in [3]. It is well known that if X is Ahlfors regular, then the inequalities in (1.2) are, in fact, equalities, see, for example [10]. Recall that a metric space X is called Ahlfors regular provided it admits a Borel regular measure μ such that

$$C^{-1} r^s \leqslant \mu\big(B(x, r)\big) \leqslant C r^s$$

E-mail address: li-jinjun@163.com.

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for some $C \geqslant 1$, for some exponent $s > 0$, and for all $x \in X, r > 0$. It is well known that self-similar sets with the open-set condition are Ahlfors regular [5]. By arguments similar to those in [5], one can prove that the graph-directed Moran fractals satisfying the open-set condition are also Ahlfors regular and therefore their Assouad dimensions equal the Hausdorff dimensions. Very recently, Olsen [8] gave a simple and direct proof that the Assouad dimension of a graph-directed Moran fractal satisfying the open-set condition coincides with its Hausdorff and box dimensions. However, in general it is difficult to obtain the Assouad dimensions of sets which are not Ahlfors regular. Mackay [7] calculated the Assouad dimension of the self-affine carpets of Bedford and McMullen and his main result solved the problem posed by Olsen [8]. In this short note, we will show that the Assouad dimensions of the Moran sets introduced by Wen [11] coincide with their packing and upper box dimensions. We would like to stress that the Moran sets discussed in this paper are different from the graph-directed Moran fractals discussed by Olsen [8]. In fact, in general the Moran sets we discussed are not Ahlfors regular.

2. Statement of results

Firstly, let us recall the definition of Moran sets introduced by Wen [11]. Let $\{n_k\}_{k \geqslant 1} \subset \mathbb{N}$ be a sequence of positive integer (we assume $n_k \geqslant 2$). For $m, k \in \mathbb{N}$, set $D_{m,k} = \{\sigma_m \sigma_{m+1} \cdots \sigma_k\colon\ 1 \leqslant \sigma_j \leqslant n_j, m \leqslant j \leqslant k\}$ and $D_k = D_{1,k}$. Define $D = \bigcup_{k=1}^{\infty} D_k$. Any element $\sigma \in D$ is called a word, by convention denoted by $D_0 = \emptyset$. If $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in D_k$ and $\tau = \tau_1 \tau_2 \cdots \tau_m \in D_{k+1,m}$, we define $\sigma * \tau = \sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_m$. For $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in D_k$, we will write $|\sigma| = k$ for the length of σ.

Suppose that $J \subset \mathbb{R}^d$ is a compact set with $\overline{\operatorname{int} J} = J$ (here and below we write $\operatorname{int} B$ and $\overline{B}$ for the interior and the closure of set B respectively). Let $\{\Phi_k\}_{k \geqslant 1}$ be a sequence of positive real vectors with $\Phi_k = (c_{k,1}, c_{k,2}, \ldots, c_{k,n_k})$, $\sum_{j=1}^{n_k} c_{k,j} \leqslant 1, k \in \mathbb{N}$. We say the collection $\mathcal{F} = \{J_\sigma\colon\ \sigma \in D\}$ of closed subsets of J fulfills the Moran structure if it satisfies the following Moran structure conditions (MSC):

(1) For $\sigma \in D$, J_σ is geometrically similar to J, i.e., there exists a similarity $S_\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that $J_\sigma = S_\sigma(J)$. For convenience we write $J_\emptyset = J$.
(2) For $k \geqslant 0$ and $\sigma \in D_k$, $J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*n_{k+1}}$ are subsets of J_σ, and satisfy that $\operatorname{int} J_{\sigma*i} \cap \operatorname{int} J_{\sigma*j} = \emptyset$ whenever $i \neq j$.
(3) For $k \geqslant 1$ and $\sigma \in D_{k-1}$,

$$\frac{|J_{\sigma*j}|}{|J_\sigma|} = c_{k,j} \quad \text{for } 1 \leqslant j \leqslant n_k,$$

where $|A|$ denotes the diameter of A.

Suppose that $\mathcal{F} = \{J_\sigma\colon\ \sigma \in D\}$ is a collection of closed subsets of J fulfilling the Moran structure. We call $E = E(\mathcal{F}) := \bigcap_{k \geqslant 1} \bigcup_{\sigma \in D_k} J_\sigma$ a Moran set determined by $\mathcal{F}$. Let $\mathcal{F}_k = \{J_\sigma\colon\ \sigma \in D_k\}$, then $\mathcal{F} = \bigcup_{k \geqslant 0} \mathcal{F}_k$. The elements of $\mathcal{F}_k$ are called kth-level basic sets of E and the elements of $\mathcal{F}$ are called the basic sets of E. Suppose the set J and the sequences $\{n_k\}, \{\Phi_k\}$ are given. We denote by $\mathcal{M} = \mathcal{M}(J, \{n_k\}, \{\Phi_k\})$ the class of the Moran sets satisfying the MSC. We call $\mathcal{M}(J, \{n_k\}, \{\Phi_k\})$ the Moran class associated with the triplet $(J, \{n_k\}, \{\Phi_k\})$.

Remark 2.1. From the above definition, we see that if the Moran sets $E_1, E_2 \in \mathcal{M}(J, \{n_k\}, \{\Phi_k\})$ and $E_1 \neq E_2$, then the relative positions of kth-level basic sets of E_1 and those of E_2 may be different, although they satisfy the same MSC.

Under some mild condition, Hua et al. [4] gave the Hausdorff, packing and upper box dimensions of Moran sets. To state their result, we need some notations. Let $\mathcal{M} = \mathcal{M}(J, \{n_k\}, \{\Phi_k\})$ be a Moran class. Let $c_* := \inf c_{i,j}$ and $c_\sigma = c_{1,\sigma_1} \cdots c_{k,\sigma_k}$ for $\sigma = \sigma_1 \cdots \sigma_k \in D_k$. Let

$$s_* = \liminf_{k \to \infty} s_k, \qquad s^* = \limsup_{k \to \infty} s_k, \tag{2.1}$$

where s_k satisfies the equation

$$\prod_{i=1}^{k} \sum_{j=1}^{n_i} c_{i,j}^{s_k} = \sum_{\sigma \in D_k} c_\sigma^{s_k} = 1. \tag{2.2}$$

We can now present the main result of Hua et al. [4].

Theorem 2.1. *(See [4].) Suppose that $\mathcal{M} = \mathcal{M}(J, \{n_k\}, \{\Phi_k\})$ is a Moran class satisfying $c_* > 0$. Then for any $E \in \mathcal{M}$,*

$$\dim_H E = s_* \quad \textit{and} \quad \dim_P E = \overline{\dim}_B E = s^*.$$

It follows from the last theorem that the Moran sets are not Ahlfors regular if $s_* \neq s^*$ and one can easily construct such ones. However, we will prove that the Assouad dimensions of the Moran sets coincide with their packing and upper box dimensions.

Theorem 2.2. *Suppose that $\mathcal{M}=\mathcal{M}(J,\{n_k\},\{\Phi_k\})$ is a Moran class satisfying $c_*>0$. Then for any $E\in\mathcal{M}$,*

$$\dim_P E=\overline{\dim}_B E=\dim_A E=s^*.$$

Remark 2.2. As we shall see, the condition $c_*>0$ plays an important role in the proof of Theorem 2.2. However, we conjecture that $\overline{\dim}_B E=\dim_A E=s^*$ remains true if the condition $c_*>0$ is removed.

3. Proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2. For $\sigma\in D$, we denote by σ^- the word obtained by deleting the last letter of σ. For $\gamma>0$, we define $\Gamma(\gamma)$ by

$$\Gamma(\gamma)=\big\{\sigma\in D\ \big|\ c_\sigma<\gamma\leqslant\sigma^-\big\}.$$

The set J contains an open ball of diameter a since it has nonempty interior. For any $\sigma\in\Gamma(\gamma)$, J_σ contains an open ball of diameter $a|J_\sigma|\geqslant ac_*\gamma$ and these open balls are disjoint by the MSC. By a standard argument (see, for example, Lemma 9.2 in [2]) we obtain the following lemma which is a generalization of a result in [5].

Lemma 3.1. *There exists a constant ℓ such that*

$$\#\big\{\sigma\in\Gamma(\gamma)\ \big|\ B(x,\gamma)\cap J_\sigma\neq\emptyset\big\}\leqslant\ell$$

for all $x\in E$ and $\gamma>0$.

Proof of Theorem 2.2. Fix $E\in\mathcal{M}$. Note the inequality (1.2); it is sufficient to prove that $\dim_A E\leqslant d$ for any $d>s^*$. By the extension theorem of measures, there exists a unique Borel probability measure μ supported on E such that

$$\mu(J_{\sigma*j})=\mu(J_\sigma)\cdot\frac{|J_{\sigma*j}|^d}{\sum_{j=1}^{n_k}|J_{\sigma*j}|^d}\tag{3.1}$$

for all $k\geqslant1,\sigma\in D_{k-1}$ and $1\leqslant j\leqslant n_k$. For $\sigma\in D$, by (3.1), we have

$$\begin{aligned}\mu(J_\sigma)&=|J_\sigma|^d\frac{|J_{\sigma^-}|^d}{|J_{\sigma^-*1}|^d+\cdots+|J_{\sigma^-*n_{|\sigma|}}|^d}\cdots\frac{|J|^d}{|J_1|^d+\cdots+|J_{n_1}|^d}\\&=|J_\sigma|^d\frac{1}{c^d_{|\sigma|,1}+\cdots+c^d_{|\sigma|,n_{|\sigma|}}}\cdots\frac{1}{c^d_{1,1}+\cdots+c^d_{1,n_1}}.\end{aligned}$$

It follows from $d>s^*$ that there exists some positive integer K such that if $k>K$, then $d>s_k$ and therefore

$$\sum_{\sigma\in D_k}c_\sigma^d=\sum_{i=1}^k\sum_{j=1}^{n_k}c_{i,j}^d\leqslant\sum_{i=1}^k\sum_{j=1}^{n_k}c_{i,j}^{s_k}=\sum_{\sigma\in D_k}c_\sigma^{s_k}=1.\tag{3.2}$$

Fix small enough $\gamma>0$. Note that $\{J_\sigma\cap E\mid\sigma\in\Gamma(\gamma)\}$ is a partition of E; we have

$$\begin{aligned}1&=\mu\Big(\bigcup_{\sigma\in\Gamma(\gamma)}J_\sigma\Big)=\sum_{\sigma\in\Gamma(\gamma)}\mu(J_\sigma)\\&=\sum_{\sigma\in\Gamma(\gamma)}|J_\sigma|^d\frac{1}{c^d_{|\sigma|,1}+\cdots+c^d_{|\sigma|,n_{|\sigma|}}}\cdots\frac{1}{c^d_{1,1}+\cdots+c^d_{1,n_1}}\\&\geqslant\sum_{\sigma\in\Gamma(\gamma)}|J|^dc_\sigma^d\cdot1\quad\big(\text{by (3.2)}\big)\\&\geqslant\sum_{\sigma\in\Gamma(\gamma)}|J|^dc^d_{\sigma^-}c^d_*\\&\geqslant\#\big(\Gamma(\gamma)\big)|J|^dc_*^d\gamma^d,\end{aligned}$$

which implies that

$$\#\big(\Gamma(\gamma)\big)\leqslant\frac{1}{|J|^dc_*^d\gamma^d}.\tag{3.3}$$

Fix $x\in E$ and $0<r<R$. For each $\sigma,\tau\in D$ choose $x_{\sigma,\tau}\in J_{\sigma*\tau}$. We claim that

$$B(x,R)\cap E\subset \bigcup_{\substack{\sigma\in\Gamma(R)\\ B(x,R)\cap J_\sigma\neq\emptyset}}\ \bigcup_{\tau\in\Gamma(\frac{r}{c_\sigma})} B(x_{\sigma,\tau},r). \tag{3.4}$$

In fact, it follows from $E\subset\bigcup_{\sigma\in\Gamma(R)}J_\sigma$ that

$$B(x,R)\cap E\subset \bigcup_{\substack{\sigma\in\Gamma(R)\\ B(x,R)\cap J_\sigma\neq\emptyset}} J_\sigma.$$

Therefore, for any $y\in B(x,R)\cap E$, we can find some $\sigma_0\in\Gamma(R)$ with $B(x,R)\cap J_{\sigma_0}\neq\emptyset$ such that $y\in J_{\sigma_0}$. Note that $J_{\sigma_0}\cap E\subset \bigcup_{\tau\in\Gamma(\frac{r}{c_{\sigma_0}})}J_{\sigma_0*\tau}$; we have

$$y\in\bigcup_{\tau\in\Gamma(\frac{r}{c_{\sigma_0}})}J_{\sigma_0*\tau},$$

and we can find some $\tau_0\in\Gamma(\frac{r}{c_{\sigma_0}})$ such that $y\in J_{\sigma_0*\tau_0}$. Clearly, $|J_{\sigma_0*\tau_0}|=c_{\sigma_0*\tau_0}\leqslant r$ since $\tau_0\in\Gamma(\frac{r}{c_{\sigma_0}})$. On the other hand, note that $x_{\sigma_0,\tau_0}\in J_{\sigma_0*\tau_0}$; we have

$$y\in J_{\sigma_0*\tau_0}\subset B(x_{\sigma_0*\tau_0},r),$$

which proves (3.4).

It follows from (3.4), (3.3) and Lemma 3.1 that

$$\begin{aligned}
N_{r,R}(E) &\leqslant \sum_{\substack{\sigma\in\Gamma(R)\\ B(x,R)\cap J_\sigma\neq\emptyset}}\ \sum_{\tau\in\Gamma(\frac{r}{c_\sigma})} 1 \quad \bigl(\text{by (3.4)}\bigr)\\
&\leqslant \sum_{\substack{\sigma\in\Gamma(R)\\ B(x,R)\cap J_\sigma\neq\emptyset}} \#\left(\Gamma\left(\frac{r}{c_\sigma}\right)\right)\\
&\leqslant \sum_{\substack{\sigma\in\Gamma(R)\\ B(x,R)\cap J_\sigma\neq\emptyset}} \frac{1}{|J|^d c_*^d}\cdot\left(\frac{c_\sigma}{r}\right)^d \quad \bigl(\text{by (3.3)}\bigr)\\
&\leqslant \sum_{\substack{\sigma\in\Gamma(R)\\ B(x,R)\cap J_\sigma\neq\emptyset}} \frac{1}{|J|^d c_*^d}\cdot\left(\frac{R}{r}\right)^d \quad \bigl(\text{since } \sigma\in\Gamma(R)\bigr)\\
&\leqslant \frac{\ell}{|J|^d c_*^d}\cdot\left(\frac{R}{r}\right)^d, \quad \bigl(\text{by Lemma (3.1)}\bigr)
\end{aligned}$$

which implies $\dim_A E\leqslant d$ for any $d>s^*$, and therefore the proof of Theorem 2.2 is completed. □

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