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On a criterion for continuity and compactness of composition operators acting on α-Bloch spaces

Sur un critère de continuité et compacité des opérateurs de composition agissant sur les espaces α-Bloch

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A B S T R A C T

In this Note we give a new criterion for the continuity and compactness of composition operators acting on α-Bloch spaces.

R É S U M É

Dans cette Note, nous donnons un nouveau critère pour la continuité et la compacité des opérateurs de composition agissant sur les espaces α-Bloch.

1. Introduction

Given a parameter $\alpha > 0$, the α-Bloch space on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, denoted as $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$, consists of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty.$$

The above relation defines a seminorm, and the α-Bloch functions form a complex Banach space $\mathcal{B}^\alpha$ with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$. When $\alpha = 1$ we get back the classical Bloch space $\mathcal{B}$.

A holomorphic function ϕ from the unit disk $\mathbb{D}$ into itself induces a linear operator C_ϕ, defined by $C_\phi(f) = f \circ \phi$, where $f \in H(\mathbb{D})$, the space of all holomorphic functions on $\mathbb{D}$, which is equipped with the topology of uniform convergence on compact subsets of $\mathbb{D}$. C_ϕ is called the *composition operator* with *symbol* ϕ. Composition operators continue to be widely studied on many subspaces of $H(\mathbb{D})$ and particularly in Bloch-type spaces.

The study of the properties of composition operators on Bloch-type spaces began with the celebrated work of Madigan and Matheson in [2], where they characterized the continuity and compactness of composition operators acting on the Bloch space $\mathcal{B}$. Many extensions of the Madigan and Matheson's results have appeared (see for instance [3] and a lot of references therein). In particular, Xiao in [8] showed the following result:

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Theorem 1.1. *(See [8].) Given $\alpha, \beta > 0$, the composition operator $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $\phi \in \mathcal{B}^\beta$ and*

$$\lim_{|\phi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\phi(z)|^2)^\alpha}\left|\phi'(z)\right| = 0.$$

Recently, many authors have found new criteria for the continuity and compactness of composition operators acting on Bloch-type spaces en terms of the n-th power of the symbol ϕ and the norm of the n-th power of the identity function on $\mathbb{D}$. The first result of this kind appears in 2009 and it is due to Wulan, Zheng, and Zhu [7], in turn, their result was extended to α-Bloch spaces by Zhao in [9]. Another criterion for the continuity and compactness of composition operators on Bloch space is due to Tjani in [5] (see also [6] or more recently [7]), she showed the following result:

Theorem 1.2. *(See [5].) The composition operator C_ϕ is compact on $\mathcal{B}$ if and only if $\phi \in \mathcal{B}$ and*

$$\lim_{|a|\to 1^-} \|\varphi_a \circ \phi\|_{\mathcal{B}} = 0,$$

where φ_a is a Möbius transformation from the unit disk onto itself; that is, $\varphi_a(z) = (a-z)/(1-\bar{a}z)$, with $z \in \mathbb{D}$.

The aim of this Note is to extend the above result to α-Bloch spaces. With this in mind, in the next section we define and study the properties of certain functions σ_a which will be the substitutes for automorphisms of the disk φ_a, thus in Section 3, we give a new criterion for the continuity of composition operators C_ϕ acting on α-Bloch spaces involving the composition of the symbol φ with the functions σ_a. Finally, in Section 4, we extend Theorem 1.2 to α-Bloch spaces.

2. Certain special functions for α-Bloch spaces

The key to our results lies in considering the following family of functions. For $a \in \mathbb{D}$ fixed, we define

$$\sigma_a(z) = \big(1-|a|\big)\big((1-\bar{a}z)^{-\alpha} - 1\big) \quad (z \in \mathbb{D}).$$

Clearly, for each $a \in \mathbb{D}$, the function σ_a has bounded derivative and for this reason we have that $\sigma_a \in \mathcal{B}^\alpha$. In fact, it is easy to see that $\sup_{a\in\mathbb{D}} \|\sigma_a\|_{\mathcal{B}^\alpha} \leqslant \alpha 2^\alpha$. Furthermore, it is clear that if $\frac{1}{2} < |a| < 1$, then

$$\left|\sigma_a'(a)\right| \geqslant \frac{\alpha}{4(1-|a|^2)^\alpha}. \tag{1}$$

Also, we can see that σ_a goes to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$.

3. Continuity

Now we can enunciate and prove our first result, this give a new criterion for the continuity of the composition operators acting on α-Bloch spaces.

Theorem 3.1. *The composition operator C_ϕ is continuous on $\mathcal{B}^\alpha$ if and only if*

$$\sup_{a\in\mathbb{D}} \|\sigma_a \circ \phi\|_\alpha < \infty. \tag{2}$$

Proof. Let us suppose first that $C_\phi : \mathcal{B}^\alpha \to \mathcal{B}^\alpha$ is continuous, then there exists a constant $M > 0$ such that $\|C_\phi(f)\|_{\mathcal{B}^\alpha} \leqslant M\|f\|_{\mathcal{B}^\alpha}$ for all $f \in \mathcal{B}^\alpha$. In particular, for each $a \in \mathbb{D}$ we have

$$\|\sigma_a \circ \phi\|_\alpha \leqslant \|\sigma_a \circ \phi\|_{\mathcal{B}^\alpha} \leqslant M\|\sigma_a\|_{\mathcal{B}^\alpha} \leqslant M\alpha 2^\alpha.$$

This shows the relation (2).

If condition (2) holds, then there exists a constant $L > 0$ such that $(1-|z|^2)^\alpha |\sigma_a'(\phi(z))||\phi'(z)| \leqslant L$ for all $a, z \in \mathbb{D}$. Taking $a = \phi(z)$ we have $(1-|z|^2)^\alpha |\sigma_{\phi(z)}'(\phi(z))||\phi'(z)| \leqslant L$ for all $z \in \mathbb{D}$. In particular, using the relation (1), we obtain

$$\frac{(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)^\alpha}\left|\phi'(z)\right| \leqslant L \tag{3}$$

whenever $\frac{1}{2} < |\phi(z)| < 1$. Furthermore, since $\frac{\alpha}{4}(\frac{1}{2})^{\alpha+1}\|\phi\|_\alpha \leqslant \|\sigma_{1/2} \circ \phi\|_\alpha \leqslant L$, then we can see that $\phi \in \mathcal{B}^\alpha$. Hence, the function on the left in the above inequality is bounded when $|\phi(z)| \leqslant \frac{1}{2}$, then we can assume that the expression (3) is valid for all $z \in \mathbb{D}$. This shows that the composition operator C_ϕ is continuous on $\mathcal{B}^\alpha$. □

4. Compactness

Now we will to extend the Tjani's result in [5] to α-Bloch spaces. The following lemma is well known and it is consequence of a more general result due to Tjani in [4]:

Lemma 4.1. *The composition operator C_ϕ is compact on $\mathcal{B}^\alpha$ if and only if given a bounded sequence $\{f_n\}$ in $\mathcal{B}^\alpha$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\phi(f_n)\|_\alpha \to 0$ as $n \to \infty$.*

As a consequence of this lemma, we can show the following result.

Theorem 4.2. *The composition operator C_ϕ is compact on $\mathcal{B}^\alpha$ if and only if $\phi \in \mathcal{B}^\alpha$ and*

$$\lim_{|a|\to 1^-} \|\sigma_a \circ \phi\|_\alpha = 0. \tag{4}$$

Proof. Suppose first that the composition operator C_ϕ is compact on $\mathcal{B}^\alpha$, let $\{a_n\}$ be any sequence in $\mathbb{D}$ such that $|a_n| \to 1$ as $n \to \infty$. Then since the sequence $\{\sigma_{a_n}\}$ is bounded in $\mathcal{B}^\alpha$ and $\sigma_{a_n} \to 0$ uniformly on compact subsets of $\mathbb{D}$, then as a consequence of Lemma 4.1 we obtain the expression (4).

Conversely, if the relation (4) holds then given $\epsilon > 0$ there exists $r_1 \in (\frac{1}{2}, 1)$ such that $\|\sigma_a \circ \phi\|_\alpha < \epsilon$ whenever $r_1 < |a| < 1$. Hence, if $z \in \mathbb{D}$ satisfies $|\phi(z)| > r_1$, then we have

$$\|\sigma_{\phi(z)} \circ \phi\|_\alpha = \sup_{w\in\mathbb{D}} \big(1-|z|^2\big)^\alpha \big|\sigma'_{\phi(z)}\big(\phi(w)\big)\big|\big|\phi'(w)\big| < \epsilon.$$

In particular, if we put $w = z$ in the above expression, we have

$$\frac{\alpha(1-|z|^2)^\alpha}{4(1-|\phi(z)|^2)^\alpha}\big|\phi'(z)\big| \leqslant \big(1-|z|^2\big)^\alpha \big|\sigma'_{\phi(z)}\big(\phi(z)\big)\big|\big|\phi'(z)\big| < \epsilon$$

whenever $|\phi(z)| > r_1$, where we have used the relation (1) in the first inequality. The result follows now by Theorem 1.1. □

Remark 1. Due to the characterizations for the compactness of composition operators on α-Bloch spaces given by us in Theorem 4.2, Xiao in [8] and by Zhao in [9], we can conclude that for $\phi \in \mathcal{B}^\alpha$, the limit in (4) and the following limits are equivalents:

(i) $\lim_{|\phi(z)|\to 1^-} (\frac{1-|z|^2}{1-|\phi(z)|^2})^\alpha |\phi'(z)| = 0$,
(ii) $\lim_{n\to\infty} n^{\alpha-1}\|\phi^n\|_\alpha = 0$.

A similar result was found by Giménez, Malavé and Ramos-Fernández in [1], but for the composition operator $C_\phi : \mathcal{B} \to \mathcal{B}^\mu$, where the weight μ can be extended to non-vanishing, complex valued holomorphic function that satisfy a reasonable geometric condition on the Euclidean disk $D(1,1)$. A function f belongs to $\mathcal{B}^\mu$ if $\|f\|_\mu = \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty$ and μ is a continuous, positive and bounded function defined on $\mathbb{D}$. We include a short and direct proof of the implication (ii) ⇒ (4) (without compactness of composition operator) for the benefit of the reader.

Suppose that the condition (ii) holds and let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n^{\alpha-1}\|\phi^n\|_\alpha < \epsilon$ for all $n \geqslant N$. Hence, we can write

$$\|\sigma_a \circ \phi\|_\alpha \leqslant \big(1-|a|\big)\sum_{n=0}^{N} b_n|a|^n \big\|\phi^n\big\|_\alpha + \epsilon\big(1-|a|\big)\sum_{n=N+1}^{+\infty} n^{1-\alpha} b_n |a|^n,$$

where b_n is the n-th coefficient of the Maclaurin serie of the function $H(w) = (1-w)^{-\alpha}$, with $|w| < 1$. Observe that the first sum on the right side of the above inequality tends to zero as $|a| \to 1^-$. Furthermore, by the Stirling's formula we have that $c_n = n^{1-\alpha} b_n \to \frac{1}{\Gamma(\alpha)} > 0$ as $n \to \infty$, hence we can ensure that there exists a constant $K_\alpha > 0$, depending only on α, such that $c_n \leqslant K_\alpha$ for all $n \in \mathbb{N}$. Thus we can conclude that $\lim_{|a|\to 1^-} \|\sigma_a \circ \phi\|_\alpha \leqslant \epsilon K_\alpha$ which shows (4).

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