Mathematical Analysis

On a criterion for continuity and compactness of composition operators acting on $\alpha$-Bloch spaces

1. Introduction

Given a parameter $\alpha > 0$, the $\alpha$-Bloch space on the unit disk $D$ of the complex plane $\mathbb{C}$, denoted as $B^\alpha = B^\alpha(D)$, consists of all holomorphic functions $f$ on $D$ such that

$$\|f\|_\alpha = \sup_{z \in D} (1 - |z|^2)\alpha |f'(z)| < \infty.$$ 

The above relation defines a seminorm, and the $\alpha$-Bloch functions form a complex Banach space $B^\alpha$ with the norm $\|f\|_{B^\alpha} = |f(0)| + \|f\|_\alpha$. When $\alpha = 1$ we get back the classical Bloch space $B$.

A holomorphic function $\phi$ from the unit disk $D$ into itself induces a linear operator $C_\phi$, defined by $C_\phi(f) = f \circ \phi$, where $f \in H(D)$, the space of all holomorphic functions on $D$, which is equipped with the topology of uniform convergence on compact subsets of $D$. $C_\phi$ is called the composition operator with symbol $\phi$. Composition operators continue to be widely studied on many subspaces of $H(D)$ and particularly in Bloch-type spaces.

The study of the properties of composition operators on Bloch-type spaces began with the celebrated work of Madigan and Matheson in [2], where they characterized the continuity and compactness of composition operators acting on the Bloch space $B$. Many extensions of the Madigan and Matheson’s results have appeared (see for instance [3] and a lot of references therein). In particular, Xiao in [8] showed the following result:

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http://dx.doi.org/10.1016/j.crma.2012.11.013
Theorem 1.1. (See [8].) Given \( \alpha, \beta > 0 \), the composition operator \( C_\phi : B^\alpha \rightarrow B^\beta \) is compact if and only if \( \phi \in B^\beta \) and
\[
\lim_{|\phi(z)| \rightarrow 1} \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| = 0.
\]

Recently, many authors have found new criteria for the continuity and compactness of composition operators acting on Bloch-type spaces (see terms of the \( n \)-th power of the symbol \( \phi \) and the norm of the \( n \)-th power of the identity function on \( \mathbb{D} \). The first result of this kind appears in 2009 and it is due to Wulan, Zheng, and Zhu [7], in turn, their result was extended to \( \alpha \)-Bloch spaces by Zhao in [9]. Another criterion for the continuity and compactness of composition operators on Bloch space is due to Tjani in [5] (see also [6] or more recently [7]), she showed the following result:

Theorem 1.2. (See [5].) The composition operator \( C_\phi \) is compact on \( B \) if and only if \( \phi \in B \) and
\[
\lim_{|a| \rightarrow 1} \| \varphi_a \circ \phi \|_B = 0,
\]
where \( \varphi_a \) is a Möbius transformation from the unit disk onto itself; that is, \( \varphi_a(z) = (a - z)/(1 - \bar{a}z) \), with \( z \in \mathbb{D} \).

The aim of this Note is to extend the above result to \( \alpha \)-Bloch spaces. With this in mind, in the next section we define and study the properties of certain functions \( \sigma_a \) which will be the substitutes for automorphisms of the disk \( \varphi_a \). Thus in Section 3, we give a new criterion for the continuity of composition operators \( C_\phi \), acting on \( \alpha \)-Bloch spaces involving the composition of the symbol \( \varphi \) with the functions \( \sigma_a \). Finally, in Section 4, we extend Theorem 1.2 to \( \alpha \)-Bloch spaces.

2. Certain special functions for \( \alpha \)-Bloch spaces

The key to our results lies in considering the following family of functions. For \( a \in \mathbb{D} \) fixed, we define
\[
\sigma_a(z) = \frac{1 - |a|}{(1 - \bar{a}z)^{-\alpha} - 1} \quad (z \in \mathbb{D}).
\]
Clearly, for each \( a \in \mathbb{D} \), the function \( \sigma_a \) has bounded derivative and for this reason we have that \( \sigma_a \in B^\alpha \). In fact, it is easy to see that \( \sup_{a \in \mathbb{D}} \| \sigma_a \|_{B^\alpha} \leq \alpha 2^\alpha \). Furthermore, it is clear that if \( \frac{1}{2} < |a| < 1 \), then
\[
\left| \sigma_a'(a) \right| \geq \frac{\alpha}{4(1 - |a|^2)\alpha}.
\]
(1)

Also, we can see that \( \sigma_a \) goes to zero uniformly on compact subsets of \( \mathbb{D} \) as \( |a| \rightarrow 1^- \).

3. Continuity

Now we can enunciate and prove our first result, this give a new criterion for the continuity of the composition operators acting on \( \alpha \)-Bloch spaces.

Theorem 3.1. The composition operator \( C_\phi \) is continuous on \( B^\alpha \) if and only if
\[
\sup_{a \in \mathbb{D}} \| \sigma_a \circ \phi \|_\alpha < \infty.
\]
(2)

Proof. Let us suppose first that \( C_\phi : B^\alpha \rightarrow B^\alpha \) is continuous, then there exists a constant \( M > 0 \) such that \( \| C_\phi (f) \|_{B^\alpha} \leq M \| f \|_{B^\alpha} \) for all \( f \in B^\alpha \). In particular, for each \( a \in \mathbb{D} \) we have
\[
\| \sigma_a \circ \phi \|_\alpha \leq \| \sigma_a \circ \phi \|_{B^\alpha} \leq M \| \sigma_a \|_{B^\alpha} \leq M \alpha 2^\alpha.
\]
This shows the relation (2).

If condition (2) holds, then there exists a constant \( L > 0 \) such that \( (1 - |z|^2)^\alpha |\sigma_a'(\phi(z))| |\phi'(z)| \leq L \) for all \( a, z \in \mathbb{D} \). Taking \( a = \phi(z) \) we have \( (1 - |z|^2)^\alpha |\sigma_\phi'(\phi(z))| |\phi'(z)| \leq L \) for all \( z \in \mathbb{D} \). In particular, using the relation (1), we obtain
\[
\frac{(1 - |z|^2)^\alpha}{(1 - |\phi(z)|^2)^\alpha} |\phi'(z)| \leq L
\]
(3)
whenever \( \frac{1}{2} < |\phi(z)| < 1 \). Furthermore, since \( \frac{3}{4} (\frac{1}{2} + 1) \| \phi \|_\alpha \leq \| \sigma_{1/2} \circ \phi \|_\alpha \leq L \), then we can see that \( \phi \in B^\alpha \). Hence, the function on the left in the above inequality is bounded when \( |\phi(z)| = \frac{1}{2} \), then we can assume that the expression (3) is valid for all \( z \in \mathbb{D} \). This shows that the composition operator \( C_\phi \) is continuous on \( B^\alpha \). \( \square \)
4. Compactness

Now we will to extend the Tjani’s result in [5] to $\alpha$-Bloch spaces. The following lemma is well known and it is consequence of a more general result due to Tjani in [4]:

Lemma 4.1. The composition operator $C_{\phi}$ is compact on $B^\alpha$ if and only if given a bounded sequence $\{f_n\}$ in $B^\alpha$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_{\phi}(f_n)\|_\alpha \to 0$ as $n \to \infty$.

As a consequence of this lemma, we can show the following result.

Theorem 4.2. The composition operator $C_{\phi}$ is compact on $B^\alpha$ if and only if $\phi \in B^\alpha$ and

$$
\lim_{|a| \to 1^-} \|\sigma_a \circ \phi\|_\alpha = 0. 
$$

Proof. Suppose first that the composition operator $C_{\phi}$ is compact on $B^\alpha$, let $\{a_n\}$ be any sequence in $\mathbb{D}$ such that $|a_n| \to 1$ as $n \to \infty$. Then since the sequence $\{\sigma_{a_n}\}$ is bounded in $B^\alpha$ and $\sigma_{a_n} \to 0$ uniformly on compact subsets of $\mathbb{D}$, then as a consequence of Lemma 4.1 we obtain the expression (4).

Conversely, if the relation (4) holds then given $\epsilon > 0$ there exists $r_1 \in (\frac{1}{2}, 1)$ such that $\|\sigma_a \circ \phi\|_\alpha < \epsilon$ whenever $r_1 < |a| < 1$. Hence, if $z \in \mathbb{D}$ satisfies $|\phi(z)| > r_1$, then we have

$$
\|\sigma_{\phi(z)} \circ \phi\|_\alpha = \sup_{w \in \mathbb{D}} (1 - |z|^2)^\alpha |\sigma_{\phi(z)}'(\phi(w))| |\phi'(w)| < \epsilon.
$$

In particular, if we put $w = z$ in the above expression, we have

$$
\frac{\alpha (1 - |z|^2)^\alpha}{4(1 - |\phi(z)|^2)^\alpha} |\phi'(z)| \leq (1 - |z|^2)^\alpha |\sigma_{\phi(z)}'(\phi(z))| |\phi'(z)| < \epsilon
$$

whenever $|\phi(z)| > r_1$, where we have used the relation (1) in the first inequality. The result follows now by Theorem 1.1.

Remark 1. Due to the characterizations for the compactness of composition operators on $\alpha$-Bloch spaces given by us in Theorem 4.2, Xiao in [8] and by Zhao in [9], we can conclude that for $\phi \in B^\alpha$, the limit in (4) and the following limits are equivalents:

(i) $\lim_{|\phi(z)| \to 1^-} (1 - |z|^2)^\alpha |\phi'(z)| = 0,$

(ii) $\lim_{n \to \infty} n^{\alpha - 1} \|\phi^n\|_\alpha = 0.$

A similar result was found by Giménez, Malavé and Ramos-Fernández in [1], but for the composition operator $C_{\phi} : B \to B^\mu$, where the weight $\mu$ can be extended to non-vanishing, complex valued holomorphic function that satisfy a reasonable geometric condition on the Euclidean disk $D(1, 1)$. A function $f$ belongs to $B^\mu$ if $\|f\|_\mu = \sup_{z \in \mathbb{D}} \mu(z)|f(z)| < \infty$ and $\mu$ is a continuous, positive and bounded function defined on $\mathbb{D}$. We include a short and direct proof of the implication (ii) $\Rightarrow$ (4) (without compactness of composition operator) for the benefit of the reader.

Suppose that the condition (ii) holds and let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n^{\alpha - 1} \|\phi^n\|_\alpha < \epsilon$ for all $n \geq N$. Hence, we can write

$$
\|\sigma_a \circ \phi\|_\alpha \leq (1 - |a|) \sum_{n=0}^{N} b_n |a|^n \|\phi^n\|_\alpha + \epsilon (1 - |a|) \sum_{n=N+1}^{+\infty} n^{1-\alpha} b_n |a|^n,
$$

where $b_n$ is the $n$-th coefficient of the Maclaurin serie of the function $H(w) = (1 - w)^{-\alpha}$, with $|w| < 1$. Observe that the first sum on the right side of the above inequality tends to zero as $|a| \to 1^-$. Furthermore, by the Stirling’s formula we have that $c_n = n^{1-\alpha} b_n \to \frac{1}{\Gamma(\alpha)} > 0$ as $n \to \infty$, hence we can ensure that there exists a constant $K_\alpha > 0$, depending only on $\alpha$, such that $c_n \leq K_\alpha$ for all $n \in \mathbb{N}$. Thus we can conclude that $\lim_{|a| \to 1^-} \|\sigma_a \circ \phi\|_\alpha \leq \epsilon K_\alpha$ which shows (4).

References