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# Complex Analysis The extendability of S-plurisubharmonic currents

### *Sur le prolongement des courants S-plurisousharmoniques*

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Article history: Received 16 June 2012 Accepted after revision 31 October 2012 Available online 21 November 2012	In this Note we are interested in studying the extension of negative S-plurisubharmonic currents across closed sets. © 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
Presented by the Editorial Board	RÉSUMÉ
	Dans cette Note, nous étudions le prolongement aux ensembles fermés des courants S- plurisousharmoniques négatifs.
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#### 1. Introduction

In complex analysis it is so familiar to study the extendability of different categories across small obstacles. Many years ago, the mathematicians started tracing the adequate conditions on their phenomena to be extended. These phenomena could be functions, analytic sets or the more general case which is currents.

Let *A* be a closed subset of an open subset  $\Omega$  of  $\mathbb{C}^n$ . For an analytic function *f* on  $\Omega \setminus A$ , Shiffman [7] (1968) proved that  $\tilde{f}$  exists, and is analytic function in the case when the Hausdorff measure  $\mathcal{H}_{2n-2}(A) = 0$ , or when  $\mathcal{H}_{2n-1}(A) = 0$  and *f* is bounded. In terms of currents, Harvey [6] (1974) studied the case when *T* is a closed positive current of bidimension (p, p) on  $\Omega \setminus A$ . For this type of currents Harvey showed that  $\tilde{T}$  exists, and is closed as soon as  $\mathcal{H}_{2p-1}(A) = 0$ .

**Definition 1.1.** Let *S* be a positive current of bidimension (p - 1, p - 1) on  $\Omega$ . A current *T* is said to be *S*-plurisubharmonic if  $dd^cT \ge -S$ . If  $dd^cT \le S$ , then *T* is called *S*-plurisuperharmonic.

In 2007, Dinh and Sibony [4] considered the case of *S*-plurisubharmonic currents and proved a version of Skoda–El Mir result for this case.

The main issue in this Note is to show the following results:

**Theorem.** (Theorem 2.2.) Let A be a closed complete pluripolar subset of an open subset  $\Omega$  of  $\mathbb{C}^n$ , and let T be a positive S-plurisuperharmonic current of bidimension (p, p) on  $\Omega \setminus A$  such that  $\mathcal{H}_{2p-1}(A \cap \overline{\text{Supp } T}) = 0$ . Then  $\widetilde{T}$  exists. Moreover, the current  $R = \widetilde{dd^cT} - dd^c\widetilde{T}$  is positive and supported in A.

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**Theorem.** (Theorem 3.1.) Let u be a positive plurisubharmonic function of class  $C^2$  on  $\Omega$  and set  $A = \{z \in \Omega; u(z) = 0\}$ . Let T be a positive S-plurisuperharmonic current of bidimension (p, p) on  $\Omega \setminus A$  such that  $\mathcal{H}_{2p-1}(A \cap \overline{\text{Supp } T}) = 0$ , and let L and K be compact sets of  $\Omega$  such that  $L \subset K^\circ$ , then there exist a constant  $C_{K,L} \ge 0$  and a neighborhood V of  $K \cap A$  which is independent of T and S such that

$$\left\| dd^{c}u \wedge T \right\|_{L \setminus A} \leq C_{K,L} \|u\|_{\mathcal{L}^{\infty}(K)} \left( \|T\|_{K \setminus V} + \left\| dd^{c}T \right\|_{K \setminus V} + \|S\|_{K} \right).$$

#### 1.1. Preparation step

Along this Note we consider that *T* is a current of bidimension (p, p) and *A* is a closed subset of an open subset  $\Omega$  of  $\mathbb{C}^n$ . By  $\widetilde{T}$  we mean the trivial extension of *T* by zero across *A*. Let us assume that  $\mathcal{H}_{2p-1}(A \cap \overline{\operatorname{Supp} T}) = 0$  and  $0 \in \overline{\operatorname{Supp} T} \cap A$ . Then by [7], there exist a system of coordinates  $(z', z'') \in \mathbb{C}^s \times \mathbb{C}^{n-s}$ , s = p - 1 and a polydisk  $\Delta^n = \Delta' \times \Delta''$  such that  $\overline{\Delta'} \times \partial \Delta'' \cap (\overline{\operatorname{Supp} T} \cap A) = \emptyset$ . Now, take 0 < t < 1 so that  $\Delta' \times \{z'', t < |z''| < 1\} \cap (\overline{\operatorname{Supp} T} \cap A) = \emptyset$ . As  $\overline{\Delta^n} \cap A$  is compact set, one can find a neighborhood  $\omega$  of  $\overline{\Delta^n} \cap A$  such that  $\omega \cap (\Delta' \times \{z'', t < |z''| < 1\}) = \emptyset$ . Let  $a \in (t, 1)$  and choose  $\rho(z') \in \mathcal{C}_0^\infty(a\Delta')$  such that  $0 \leq \rho \leq 1$  and  $\rho = 1$  on  $\frac{1}{2}a\Delta'$ . Take  $\chi \in \mathcal{C}_0^\infty(\omega)$  such that  $0 \leq \chi \leq 1$  and  $\chi = 1$  on a neighborhood  $\omega_0$  of  $\overline{\Delta^n} \cap A$ . Obviously, the function  $\chi(z)\rho(z')$  is positive smooth compactly supported in  $a\Delta^n$ . For convenience, we set  $\beta' = dd^c(|z'|^2)$ ,  $\beta'' = dd^c(|z''|^2)$  and  $\alpha(z') = \rho(z')\beta'^s$ . By  $\max_{\varepsilon}(x_1, x_2)$ , we mean the function  $\max_{\varepsilon}(x_1, x_2) = \max(x_1, x_2) * \alpha_{\varepsilon}$ , where  $\alpha_{\varepsilon}$  is a regularization kernel on  $\mathbb{R}^2$  depending only on  $\|(x_1, x_2)\|$ .

#### 2. The case when A is a pluripolar set

**Lemma 2.1.** Let A be a closed complete pluripolar subset of  $\Omega$  and T be a positive S-plurisuperharmonic current of bidimension (p, p)on  $\Omega \setminus A$ . Let v be a plurisubharmonic function of class  $C^2$ ,  $v \ge -1$  on  $\Omega$  such that  $\Omega' = \{z \in \Omega : v(z) < 0\}$  is relatively compact in  $\Omega$ . Let  $K \subset \Omega'$  be a compact subset and set  $c_K = -\sup_{z \in K} v(z)$ . Then there exists a constant  $\eta \ge 0$  such that for every plurisubharmonic function u on  $\Omega'$  of class  $C^2$  satisfying  $-1 \le u < 0$  we have

$$\int_{K\setminus A} T \wedge dd^{c} u \wedge \beta^{p-1} \leqslant \frac{1}{c_{K}} \int_{\Omega'\setminus A} T \wedge dd^{c} v \wedge \beta^{p-1} + \eta \|S\|_{\Omega'}.$$
(1)

**Proof.** We follow the same techniques as in [3]. By [5, Proposition II.2] there exists a negative plurisubharmonic function f on  $\Omega'$  which is smooth on  $\Omega' \setminus A$  such that  $A \cap \Omega' = \{z \in \Omega': f(z) = -\infty\}$ . We choose  $\lambda, \mu$  such that  $0 < \mu < \lambda < c_K$ . For  $m \in \mathbb{N}$  and  $\varepsilon$  small enough we set

$$\varphi_m(z) = \mu u(z) + \frac{f(z) + m}{m+1} \quad \text{and} \quad \varphi_{m,\varepsilon}(z) = \max_{\varepsilon} \left( v(z) + 1, \varphi_m(z) \right). \tag{2}$$

Thus,  $\varphi_{m,\varepsilon} \in Psh(\Omega') \cap C^{\infty}(\Omega')$ . Furthermore,  $\varphi_{m,\varepsilon}(z) = v(z) + 1$  on a neighborhood of  $\partial \Omega' \cup (\Omega' \cap \{f \leq -m\})$ . Consider the open subset  $\Omega'_m = \Omega' \cap \{f > -m\}$ . Then by Stokes' formula we have

$$\int_{\Omega'_m} T \wedge dd^c (\varphi_{m,\varepsilon} - \nu - 1) \wedge \beta^{p-1} \leqslant \int_{\Omega'_m} (\varphi_{m,\varepsilon} - \nu - 1) S \wedge \beta^{p-1}.$$

Hence, there exists a constant  $\eta > 0$  so that

$$\int_{\Omega'_{m}} T \wedge dd^{c} \varphi_{m,\varepsilon} \wedge \beta^{p-1} \leqslant \int_{\Omega'_{m}} T \wedge dd^{c} \nu \wedge \beta^{p-1} + \eta \|S\|_{\Omega'}.$$
(3)

Let R > 0 and  $K_R = \{z \in K: f(z) \ge -R\}$ . For *m* sufficiently large, we have  $K_R \subset \Omega'_m$  and  $\varphi_m(z) \ge -\mu + \frac{m-R}{m+1} > 1 - \lambda$  for any  $z \in K_R$ . Since  $v \le -c_K$  on  $K_R$ , then  $v + 1 \le 1 - c_K \le 1 - \lambda$ , and because of this we find that  $\varphi_{m,\varepsilon} = \varphi_m$  on a neighborhood of  $K_R$ . Therefore, by (3) we get

$$\int_{K_R} T \wedge dd^c \varphi_m \wedge \beta^{p-1} \leqslant \int_{\Omega'_m} T \wedge dd^c \nu \wedge \beta^{p-1} + \eta \|S\|_{\Omega'}.$$
(4)

As  $dd^c f \ge 0$ , the form  $dd^c \varphi_m \ge \mu dd^c u$ , thus

$$\mu \int_{K_R} T \wedge dd^c u \wedge \beta^{p-1} \leqslant \int_{\Omega' \setminus A} T \wedge dd^c v \wedge \beta^{p-1} + \eta \|S\|_{\Omega'}.$$
(5)

The result now follows by letting first  $R \to \infty$  and  $\mu \to c_K$ .  $\Box$ 

**Theorem 2.2.** Let A be a closed complete pluripolar subset of  $\Omega$ , and let T be a positive S-plurisuperharmonic current of bidimension (p, p) on  $\Omega \setminus A$  such that  $\mathcal{H}_{2p-1}(A \cap \overline{\text{Supp }T}) = 0$ . Then  $\tilde{T}$  exists. Moreover, the current  $R = dd^c T - dd^c \tilde{T}$  is positive and supported in A.

**Proof.** Let us first assume that  $\tilde{T}$  exists. Then by [4, Theorem 1.3] the extension  $dd^cT$  exists and R is positive current. Now, we show the existence of  $\tilde{T}$ . As our problem is local and  $\mathcal{H}_{2p-1}(A \cap \overline{\text{Supp }T}) = 0$ , we keep the choices of the preparation step. For small  $\varepsilon_1 > 0$ , set

$$\Gamma_{\varepsilon_1} = \max_{\varepsilon_1} \left( \frac{|z'|^2 - a^2}{a^2}, \frac{|z''|^2 - a^2}{a^2 - t^2} \right).$$
(6)

Obviously, the function  $\Gamma_{\varepsilon_1}$  is smooth plurisubharmonic satisfying

$$-1 \leqslant \Gamma_{\varepsilon_1} < 0 \quad \text{in } a\Delta^n \quad \text{and} \quad \Gamma_{\varepsilon_1} = \frac{|z'|^2 - a^2}{a^2} \quad \text{on } \{ |z''| < t \}.$$

$$\tag{7}$$

Furthermore, the current  $T \wedge dd^c \Gamma_{\varepsilon_1} \wedge \beta'^s$  is defined on  $a\Delta^n$ , and by [4, Lemma 2.2] there exists a constant C' > 0 such that

$$\int_{a\Delta^{n}} T \wedge dd^{c} \Gamma_{\varepsilon_{1}} \wedge \beta'^{s} = \int_{a\Delta' \times \{|z''| \ge t\}} T \wedge dd^{c} \Gamma_{\varepsilon_{1}} \wedge \beta'^{s} \leqslant C' \big( \|T\|_{a\Delta^{n} \setminus V} + \|dd^{c}T\|_{a\Delta^{n} \setminus V} \big) < \infty,$$
(8)

for some neighborhood V of  $a \Delta^n \cap A$ . By applying Lemma 2.1 we deduce that

$$\int_{t\Delta^n\setminus A} T\wedge\beta\wedge\beta'^{s}<\infty.$$
(9)

Hence  $\tilde{T}$  exists since (9) is fulfilled for almost all choices of unitary coordinates (z', z'').

**Corollary 2.3.** Under the assumptions of Theorem 2.2, if A is an analytic set, dim A < p and S is closed, then  $R = \sum_{j} h_j [A_j]$  where  $\{A_j\}$  are the irreducible components of A of maximal dimensions and  $h_j$  are positive constants.

**Proof.** By Skoda–El Mir we have that  $dd^cT$  is closed, and so is the current *R*. Now, the conclusion follows from [2, Theorem 2.1].  $\Box$ 

#### 3. The case when A is a zero set of plurisubharmonic function

This section is devoted to study the continuation over zero sets of plurisubharmonic functions. So, let u be a nonnegative plurisubharmonic function on  $\Omega$  of class  $C^2$ , and set  $A = \{z \in \Omega; u(z) = 0\}$ .

**Theorem 3.1.** Let *T* be a positive *S*-plurisuperharmonic current of bidimension (p, p) on  $\Omega \setminus A$  such that  $\mathcal{H}_{2p-1}(A \cap \overline{\text{Supp }T}) = 0$ . Let *L* and *K* be compact sets of  $\Omega$  such that  $L \subset K^\circ$ , then there exist a constant  $C_{K,L} \ge 0$  and a neighborhood *V* of  $K \cap A$  which is independent of *T* and *S* such that

$$\left\| dd^{c}u \wedge T \right\|_{L \setminus A} \leq C_{K,L} \|u\|_{\mathcal{L}^{\infty}(K)} \left( \|T\|_{K \setminus V} + \left\| dd^{c}T \right\|_{K \setminus V} + \|S\|_{K} \right).$$

**Proof.** Suppose that  $0 \le u \le 1$ , and  $0 \in \overline{\text{Supp }T} \cap A$ . Since  $\mathcal{H}_{2p-1}(A \cap \overline{\text{Supp }T}) = 0$ , we will use the notations of the preparation step. Without loss of generality we may assume that  $\overline{\Delta^n} \subset K$  and  $\omega$  is a neighborhood of  $K \cap A$  such that  $\omega \cap (\Delta' \times \{t < |z''| < 1\}) = \emptyset$ . Let us set  $\Omega_c = \{z, u(z) \le c\}$ , and for small  $\varepsilon_k > 0$  put  $u_k = \max(u - \frac{1}{k}, 0) * \chi_{\varepsilon_k}$ , where  $\chi_{\varepsilon}$  is a regularization kernel on  $\mathbb{C}^n$  depending only on |z|. Notice that  $u_k = 0$  in  $\Delta^n \cap \Omega_{\frac{1}{2^2}}$ , and on  $\Delta^n \setminus \Omega_{\frac{2}{k}}$ ;  $dd^c u_k \ge \frac{1}{2} dd^c u$ . Hence

$$\int_{X'\times\Delta''} dd^c(\chi u_k)\wedge T\wedge\alpha(z') \leqslant \int_{\Delta'\times\Delta''} \chi u_k S\wedge\alpha(z') \leqslant \|S\|_K.$$
(10)

This implies that

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$$\int_{\Delta' \times \Delta''} \chi(z) dd^{c} u_{k} \wedge T \wedge \alpha(z')$$

$$\leq \|S\|_{K} + \left| \int_{\Delta' \times \Delta''} u_{k} dd^{c} \chi \wedge T \wedge \alpha(z') \right| + \left| \int_{\Delta' \times \Delta''} du_{k} \wedge d^{c} \chi \wedge T \wedge \alpha(z') \right| + \left| \int_{\Delta' \times \Delta''} d\chi \wedge d^{c} u_{k} \wedge T \wedge \alpha(z') \right|.$$
(11)

The forms  $d\chi$ ,  $d^c\chi$  and  $dd^c\chi$  vanish on some neighborhood V' of  $K \cap A$ , therefore we can change  $\Delta' \times \Delta''$  by  $\Delta' \times \Delta'' \setminus V'$ in the right-hand side integrals of (11). Now, take V a neighborhood of  $K \cap A$  so that  $\overline{V} \subset V'$ . Then by Cauchy–Schwartz inequality and [1, Lemma 3.1], there exists a constant  $C''_{K,L} > 0$  such that

$$\int_{\Delta' \times \Delta''} \chi(z) dd^c u_k \wedge T \wedge \alpha(z') \leq C_{K,L}''(\|T\|_{K \setminus V} + \|dd^c T\|_{K \setminus V} + \|S\|_K).$$
(12)

But the assumptions of  $u_k$  gives that

$$\int_{(\frac{1}{2}a\Delta'\times\Delta'')\cap\omega_0\setminus\Omega_{\frac{2}{k}}} dd^c u\wedge T\wedge\beta'^s \leq 2\int_{\Delta'\times\Delta''}\chi(z)dd^c u_k\wedge T\wedge\alpha(z').$$

So, by letting *k* tends to  $\infty$ , we obtain

$$\int_{(\frac{1}{2}a\Delta'\times\Delta'')\cap\omega_0\setminus A} T\wedge dd^c u\wedge\beta'^s \leqslant C_{K,L}'(\|T\|_{K\setminus V}+\|dd^c T\|_{K\setminus V}+\|S\|_K).$$
(13)

Notice that, what we have shown in (13) is true for almost all choices of unitary coordinates (z', z''), thus

$$\left\| dd^{c}u \wedge T \right\|_{L \setminus A} \leq C_{K,L} \left( \|T\|_{K \setminus V} + \left\| dd^{c}T \right\|_{K \setminus V} + \|S\|_{K} \right).$$

$$\tag{14}$$

Now, for any nonnegative plurisubharmonic function u of class  $C^2$ , we take

$$u^*=\frac{u}{\|u\|_{\mathcal{L}^\infty(K)}},$$

and the result follows from (14).  $\Box$ 

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