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Sampling in a weighted Sobolev space

Échantillonage dans un espace de Sobolev avec poids

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ABSTRACT

We show that functions f in some weighted Sobolev space are completely determined by time-frequency samples $\{f(t_n)\}_{n\in\mathbb{Z}} \cup \{\hat{f}(\lambda_k)\}_{k\in\mathbb{Z}}$ along appropriate slowly increasing sequences $\{t_n\}_{n\in\mathbb{Z}}$ and $\{\lambda_n\}_{n\in\mathbb{Z}}$ tending to $\pm\infty$ as $n \to \pm\infty$.

RÉSUMÉ

Nous démontrons que toute fonction f dans un certain espace de Sobolev avec poids est complètement determinée par un échantillon $\{f(t_n)\}_{n\in\mathbb{Z}} \cup \{\hat{f}(\lambda_k)\}_{k\in\mathbb{Z}}$ sur des convenables suites croissantes $\{t_n\}_{n\in\mathbb{Z}}$ et $\{\lambda_n\}_{n\in\mathbb{Z}}$, tendant vers $\pm\infty$ lentement, quand $n \to \pm\infty$.

1. Introduction and notations

If $0 \neq x \in \mathbb{R}^N$ and $\hat{x}$ denote its discrete Fourier transform, then $l(x) \cdot l(\hat{x}) \geqslant N$, where $l(x)$ denotes the cardinality of $\{k\colon\ x_k \neq 0\}$ [3]. Another result of the same essence appears in [9]: if a nonzero function is bandlimited to $[-\Omega, \Omega]$, then there exists an interval of length greater than π/Ω on which the function does not vanish.

Our objective is to extend this form of the uncertainty principle to a weighted Sobolev space. Given functions φ and ψ such that $\varphi(t) \geqslant t^2$, $\psi(t) \geqslant t^2$, let $\mathcal{H}_{\varphi,\psi}$ denote the Hilbert space of functions $f \in L^2(\mathbb{R})$ such that

$$\|f\|^2 = \int_{\mathbb{R}} \left(\left|f(t)\right|^2 \varphi(t) + \left|\hat{f}(t)\right|^2 \psi(t)\right) \mathrm{d}t < \infty \tag{1}$$

where $\hat{f}$ is the Fourier transform of f. First, we show that if $f \in \mathcal{H}_{\varphi,\psi}$ and f and $\hat{f}$ are zero respectively, on slowly increasing sequences $\{t_n\}_{n\in\mathbb{Z}}$ and $\{\lambda_n\}_{n\in\mathbb{Z}}$ tending to $\pm\infty$ as $n \to \pm\infty$, then $f \equiv 0$. We then introduce an equivalent discrete norm on $\mathcal{H}$, in terms of pointwise time-frequency samples $\{f(t_n)\}_{n\in\mathbb{Z}} \cup \{\hat{f}(\lambda_k)\}_{k\in\mathbb{Z}}$. The Riesz Representation Theorem shows how $f \in \mathcal{H}$ can be reconstructed from these time-frequency samples. A special case of this is a Poisson summation formula on slowly increasing sequences. Finally, we show that the weights $\varphi(t) = t^2 = \psi(t)$ are optimal.

We recall that reconstruction of bandlimited signals f from pointwise samples $\{f(t_n)\}_{n\in\mathbb{Z}}$ has been widely studied [1,2, 4–7].

Notations: Given an increasing sequence $\mathcal{T} = \{t_k\}_{k\in\mathbb{Z}}$ in $\mathbb{R}$ such that $\lim_{n\to\pm\infty} t_n = \pm\infty$, we define the sampling operator $S_{\mathcal{T}}$ by $S_{\mathcal{T}} g = \sum_{k\in\mathbb{Z}} g(t_k) 1_{T_k}$ where $T_k = [\frac{1}{2}(t_{k-1} + t_k), \frac{1}{2}(t_k + t_{k+1})]$. Moreover, given a non-negative function φ on $\mathbb{R}$, we

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introduce a measure $D_{\mathcal{T},\varphi}$ of the density of the sequence $\mathcal{T}$ by $D_{\mathcal{T},\varphi} = \sup_{k\in\mathbb{Z}} \int_{T_k} |t - t_k| \cdot \varphi(t)\,dt$. If $\varphi(t) = |t|^p$, we shall simply write $D_{\mathcal{T},p}$ in place of $D_{\mathcal{T},\varphi}$. We also define a weighted energy $E_\varphi(f)$ of a signal f by $E_\varphi(f) = \int_{\mathbb{R}} |f(t)|^2\varphi(t)\,dt$. Finally, we define the Fourier transform of a function $f \in L^1(\mathbb{R})$ by $\hat{f}(w) = (2\pi)^{-1/2}\int_{\mathbb{R}} f(t)e^{-iwt}\,dt$.

2. Time-frequency zeros of functions in $\mathcal{H}_{\varphi,\psi}$

In this note, $\mathcal{T} = \{t_k\}_{k\in\mathbb{Z}}$ and $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ will always denote increasing sequences of real numbers such that $\lim_{n\to\pm\infty} t_n = \pm\infty = \lim_{n\to\pm\infty}\lambda_n$. Also, φ and ψ will always denote functions defined on $\mathbb{R}$ satisfying $\varphi(t) \geqslant t^2$ and $\psi(t) \geqslant t^2$, for $t \in \mathbb{R}$. Recall the Hilbert space $\mathcal{H}_{\varphi,\psi}$ defined in (1).

We begin with a basic estimate with the sampling operator $S_{\mathcal{T}}$.

Lemma 2.1. *Let $f \in \mathcal{H}_{\varphi,\psi}$. Then $E_\varphi(f - S_{\mathcal{T}} f) \leqslant D_{\mathcal{T},\varphi}\|f'\|_2^2$.*

Proof. Given $t \in T_k$, $|f(t) - f(t_k)|^2 \leqslant |t - t_k| \int_{T_k} |f'(s)|^2\,ds$. Combining this with the identity $E_\varphi(f - S_{\mathcal{T}}(f)) = \sum_{n\in\mathbb{Z}} \int_{T_n} |f(t) - f(t_n)|^2\varphi(t)\,dt$, yields the conclusion. □

The next theorem is the main result of this section.

Theorem 2.2. *Let $f \in \mathcal{H}_{\varphi,\psi}$ such that $f(t_k) = 0 = \hat{f}(\lambda_k)$ for each $k \in \mathbb{Z}$. If $D_{\mathcal{T},\varphi} \cdot D_{\Lambda,\psi} < 1$, then $f \equiv 0$.*

Proof. Lemma 2.1 implies $E_\varphi(f) \leqslant D_{\mathcal{T},\varphi}\|f'\|_2^2$ and $E_\psi(\hat{f}) \leqslant D_{\Lambda,\psi}\|\hat{f}'\|_2^2$. Combining this with $\|f'\|_2^2 = \int_{\mathbb{R}} |\xi\hat{f}(\xi)|^2\,d\xi \leqslant E_\psi(\hat{f})$, yields $E_\varphi(f) \leqslant D_{\mathcal{T},\varphi} \cdot D_{\Lambda,\psi}\|\hat{f}'\|_2^2$. In view of $\|\hat{f}'\|_2^2 = \int_{\mathbb{R}} |tf(t)|^2\,dt \leqslant E_\varphi(f)$, we see that $E_\varphi(f) \leqslant D_{\mathcal{T},\varphi} \cdot D_{\Lambda,\psi} \cdot E_\varphi(f)$. If $f \not\equiv 0$, then $1 \leqslant D_{\mathcal{T},\varphi} \cdot D_{\Lambda,\psi}$, a contradiction. □

Example 2.3. For each $n \in \mathbb{Z}$, define $t_{\pm n} = \pm\ln(|n| + 1)$. Let $\mathcal{T} = \{t_k\}_{k\in\mathbb{Z}}$. Then for any $p > 0$, $D_{\mathcal{T},p} < \infty$.

Example 2.4. Let α and p be positive numbers such that $\alpha(p + 2) < 2$. Let $t_{\pm n} = \pm|n|^\alpha$ for $n \in \mathbb{Z}$. Let $\mathcal{T} = \{t_k\}_{k\in\mathbb{Z}}$. Then $D_{\mathcal{T},p} < \infty$.

Remark 2.5. Let $\{t_n\}_{n\in\mathbb{Z}}$ be any increasing sequence such that $\lim_{n\to\pm\infty} t_n = \pm\infty$. Let $\mathcal{T} = \{t_k\}_{k\in\mathbb{Z}}$ and $\mathcal{T}_\varepsilon = \{\varepsilon t_k\}_{k\in\mathbb{Z}}$. Then $D_{\mathcal{T}_\varepsilon,p} = \varepsilon^{2+p} D_{\mathcal{T},p}$ for any $p > 0$.

3. Equivalent discrete norms on $\mathcal{H}_{\varphi,\psi}$

Our main result, Theorem 3.3, gives equivalent norms on the Hilbert space $\mathcal{H}_{\varphi,\psi}$. First, we state the following lemma, omitting its proof:

Lemma 3.1. *Given measurable functions $f, g : \mathbb{R} \to \mathbb{C}$, $\frac{1}{2}E_\varphi(f - g) \leqslant E_\varphi(f) + E_\varphi(g)$.*

Proposition 3.2. *Suppose φ is even and $t_{-k} = -t_k$ for each $k \in \mathbb{Z}$. Let $f \in \mathcal{H}_{\varphi,\psi}$. With $C_0 = 1 - 4D_{\mathcal{T},\varphi}D_{\Lambda,\psi}$, we have*

$$C_0 \cdot E_\varphi(f) \leqslant 4D_{\mathcal{T},\varphi} \cdot E_\psi(S_\Lambda \hat{f}) + 2E_\varphi(S_{\mathcal{T}} f), \tag{2}$$

$$C_0 \cdot E_\psi(\hat{f}) \leqslant 4D_{\Lambda,\psi} \cdot E_\varphi(S_{\mathcal{T}} f) + 2E_\psi(S_\Lambda \hat{f}). \tag{3}$$

Proof. From $\|f'\|_2^2 \leqslant E_\psi(\hat{f})$, we obtain $E_\varphi(f) \leqslant 2D_{\mathcal{T},\varphi}E_\psi(\hat{f}) + 2E_\varphi(S_{\mathcal{T}} f)$ in view of Lemmas 2.1 and 3.1. Replacing $(f, \varphi, \psi, \mathcal{T})$ by $(\hat{f}, \psi, \varphi, \Lambda)$ in this inequality and noting that the Fourier transform of $\hat{f}$ is the function $\tilde{f}(x) = f(-x)$ yields $E_\psi(\hat{f}) \leqslant 2D_{\Lambda,\psi}E_\varphi(\tilde{f}) + 2E_\psi(S_\Lambda \hat{f})$. Since φ is even, $E_\varphi(\tilde{f}) = E_\varphi(f)$. Thus, combining the last two inequalities gives the estimate (2).

Now, the assumptions that φ is even and $t_{-k} = -t_k$ imply $E_\varphi(S_{\mathcal{T}}\tilde{f}) = E_\varphi(S_{\mathcal{T}} f)$. Thus, applying (2) with $(f, \varphi, \psi, \mathcal{T}, \Lambda)$ replaced by $(\hat{f}, \psi, \varphi, \Lambda, \mathcal{T})$ yields (3). □

We come to our main theorem.

Theorem 3.3. *Suppose φ is even and $t_{-k} = -t_k$ if $k \in \mathbb{Z}$. Let $4D_{\mathcal{T},\varphi}D_{\Lambda,\psi} < 1$ and $f \in \mathcal{H}_{\varphi,\psi}$. Then*

$$C_1\|f\|_\star^2 \leqslant \|f\|^2 \leqslant C_2\|f\|_\star^2 \tag{4}$$

where $\|\cdot\|^2$ *is defined in* (1), $\|f\|_\star^2 = \sum_{k\in\mathbb{Z}}(|f(t_k)|^2\int_{T_k}\varphi(t)\,\mathrm{d}t + |\hat{f}(\lambda_k)|^2\int_{\Lambda_k}\psi(t)\,\mathrm{d}t)$, $C_1 = (2+2\max\{D_{\Lambda,\psi}, D_{\mathcal{T},\varphi}\})^{-1}$, *and* $C_2 = (2+4\max\{D_{\Lambda,\psi}, D_{\mathcal{T},\varphi}\})(1-4D_{\mathcal{T},\varphi}D_{\Lambda,\psi})^{-1}$.

Proof. Adding (2) and (3) gives $\|f\|^2 \leqslant C_2\|f\|_\star^2$. By Lemmas 2.1 and 3.1 and the bound, $\|f'\|_2^2 \leqslant E_\psi(\hat{f})$, we get $E_\varphi(S_\mathcal{T} f) \leqslant 2D_{\mathcal{T},\varphi}E_\psi(\hat{f}) + 2E_\varphi(f)$. Since φ is even, we likewise obtain $E_\psi(S_\Lambda \hat{f}) \leqslant 2D_{\Lambda,\psi}E_\varphi(f) + 2E_\psi(\hat{f})$. Adding these last two inequalities gives $C_1\|f\|_\star^2 \leqslant \|f\|^2$. □

4. Time-frequency expansions and a Poisson summation formula

An application of Theorem 3.3 gives an expansion of $f \in \mathcal{H}_{\varphi,\psi}$ in terms of time-frequency samples $\{f(t_n)\}_{n\in\mathbb{Z}} \cup \{\hat{f}(\lambda_k)\}_{k\in\mathbb{Z}}$.

Corollary 4.1. *Let* $4D_{\mathcal{T},\varphi}D_{\Lambda,\psi} < 1$. *Suppose* φ *is even and* $t_{-k} = -t_k$ *for each* $k \in \mathbb{Z}$.

(a) *Given* $x \in \mathbb{R}$, *there exists* $\Phi_x \in \mathcal{H}_{\varphi,\psi}$ *such that for each* $f \in \mathcal{H}_{\varphi,\psi}$,

$$f(x) = \sum_{k\in\mathbb{Z}}\left(f(t_k)\overline{\Phi_x(t_k)}\int_{T_k}\varphi(t)\,\mathrm{d}t + \hat{f}(\lambda_k)\overline{\widehat{\Phi_x}(\lambda_k)}\int_{\Lambda_k}\psi(t)\,\mathrm{d}t\right). \tag{5}$$

(b) *For* $k \in \mathbb{Z}$, *set* $a_k = \overline{\Phi_0(t_k)}\int_{T_k}\varphi(t)\,\mathrm{d}t$ *and* $b_k = \overline{\widehat{\Phi_0}(\lambda_k)}\int_{\Lambda_k}\psi(t)\,\mathrm{d}t$. *Then*

$$f(0) = \sum_{k\in\mathbb{Z}}\big(a_k f(t_k) + b_k\hat{f}(\lambda_k)\big), \quad \text{if } f \in \mathcal{H}_{\varphi,\psi}. \tag{6}$$

Proof. Let $x \in \mathbb{R}$. Then the mapping $f \mapsto f(x)$ defines a bounded linear functional on $\mathcal{H}_{\varphi,\psi}$. Note that the inner product

$$\langle f, g\rangle_\star = \sum_{k\in\mathbb{Z}}\left(f(t_k)\overline{g(t_k)}\int_{T_k}\varphi(t)\,\mathrm{d}t + \hat{f}(\lambda_k)\overline{\hat{g}(\lambda_k)}\int_{\Lambda_k}\psi(t)\,\mathrm{d}t\right)$$

induces the equivalent norm $\|\cdot\|_\star$ on the Hilbert space $\mathcal{H}_{\varphi,\psi}$, by Theorem 3.3. Thus, the Riesz Representation Theorem implies the existence of $\Phi_x \in \mathcal{H}_{\varphi,\psi}$ such that $f(x) = \langle f, \Phi_x\rangle_\star$ for each $f \in \mathcal{H}_{\varphi,\psi}$. This is the desired conclusion (5). Part (b) is obtained by taking $x = 0$ in (5). □

5. Optimality of the weight t^2

We end this note with Example 5.2, showing optimality of the weights $\varphi(t) = t^2 = \psi(t)$.

Lemma 5.1. *Let* $H(x) = e^{-|x|}$, $T(x) = \cos(\alpha x^2)$, *and set* $f = H \star T$. *Then*

(a) f *has a zero on the interval* $[\operatorname{sgn} k\sqrt{\frac{|k|\pi}{\alpha}}, \operatorname{sgn} k\sqrt{\frac{|k+1|\pi}{\alpha}}]$ *for each* $k \in \mathbb{Z}$, *with* $\operatorname{sgn} 0 = 1$,

(b) $\hat{f}$ *is zero at* $\operatorname{sgn} k\sqrt{\alpha\pi(4|k|-1)}$ *for each* $k \in \mathbb{Z}\setminus\{0\}$.

Proof. Set $x_l = \operatorname{sgn} l\sqrt{\frac{|l|\pi}{\alpha}}$ for $l \in \mathbb{Z}$. Fix $k \in \mathbb{Z}$. We claim that $(-1)^k f(x_k) > 0$.

We shall only prove the case when $k \geqslant 0$, which we now assume. Let $l \geqslant k$ such that $l-k$ is even. Integration by parts gives

$$\int_{x_l}^{x_{l+2}} \cos(\alpha x^2)\exp\{-(x-x_k)\}\,\mathrm{d}x = R_l\exp\{-(x_{l+2}-x_k)\} + S_l \quad \text{with} \tag{7}$$

$$4\sqrt{\alpha}R_l = \int_{l\pi}^{(l+1)\pi}\{x^{-3/2} - (x+\pi)^{-3/2}\}\sin x\,\mathrm{d}x, \tag{8}$$

$$4\alpha S_l = \int_{l\pi}^{(l+1)\pi}\big(G(y) - G(y+\pi)\big)\sin y\,\mathrm{d}y, \tag{9}$$

$$G(y) = y^{-1/2} g(y) + \frac{1}{2} y^{-3/2} \int_y^{(l+2)\pi} g(t)\,\mathrm{d}t \quad \text{and} \quad g(t) = t^{-1/2} \exp\left\{-\left(\sqrt{\frac{t}{\alpha}} - x_k\right)\right\}.$$

In view of (8), $(-1)^k R_l = (-1)^l R_l > 0$. On the other hand, since g is a non-negative decreasing function on $[k\pi, \infty[$, the same is true of G. Thus, (9) shows that $(-1)^k S_l = (-1)^l S_l > 0$.

Summing (7), we conclude that $(-1)^k \int_{x_k}^{\infty} \cos(\alpha x^2) \exp(-|x - x_k|)\,\mathrm{d}x > 0$. Likewise, we have

$$(-1)^k \int_{-\infty}^{x_k} \cos(\alpha x^2) \exp(-|x - x_k|)\,\mathrm{d}x > 0.$$

Adding these yields $(-1)^k f(x_k) > 0$.

To prove (b), recall that $2\sqrt{2\alpha} \int_0^{\infty} \cos(\alpha x^2) \cos(tx)\,\mathrm{d}x = \sqrt{\pi}[\cos(\frac{t^2}{4\alpha}) + \sin(\frac{t^2}{4\alpha})]$. (See [8].) Thus, $\sqrt{2\alpha}\hat{T}(w) = \sin(\frac{w^2}{4\alpha} + \frac{\pi}{4})$. Part (b) follows from this since $\hat{f} = \hat{H} \cdot \hat{T}$. □

Recall that if $\varphi(t) = |t|^p$, we write $D_{\mathcal{T},p}$ in place of $D_{\mathcal{T},\varphi}$.

Example 5.2. Given $p \in [1, 2[$, there exist increasing sequences $\mathcal{T} = \{t_k\}_{k\in\mathbb{Z}}$ and $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$ such that $\lim_{n\to\pm\infty} t_n = \pm\infty = \lim_{n\to\pm\infty} \lambda_n$ and $D_{\mathcal{T},p} D_{\Lambda,2} < 1$, but $f(t_k) = 0 = \hat{f}(\lambda_k)$, $\forall k \in \mathbb{Z}$, for some nonzero $f \in L^2(\mathbb{R})$ with $\int_{\mathbb{R}} (|f(x)|^2 + |\hat{f}(x)|^2) x^2\,\mathrm{d}x < \infty$.

Proof. Let $p \in [1, 2[$ and choose $\alpha > 0$ such that $25\pi^2 (2\pi)^{1+\frac{p}{2}} \alpha^{1-\frac{p}{2}} < 1$. As in Lemma 5.1, we define $f = H \star T$. Recall that $\sqrt{\pi}\hat{H}(w) = \sqrt{2}(1 + w^2)^{-1}$ and $\sqrt{2\alpha}\hat{T}(w) = \sin(\frac{w^2}{4\alpha} + \frac{\pi}{4})$. Thus, $\hat{f}(w) = (\pi\alpha)^{-1/2}(1 + w^2)^{-1} \sin(\frac{w^2}{4\alpha} + \frac{\pi}{4})$. It follows that $\int_{\mathbb{R}} |\hat{f}(w)|^2 |w|^2\,\mathrm{d}w < \infty$ and $\int_{\mathbb{R}} |f(x)|^2 x^2\,\mathrm{d}x = \int_{\mathbb{R}} |\hat{f}'(w)|^2\,\mathrm{d}w < \infty$.

For $k \in \mathbb{Z}$, f has a zero t_k in the interval $[\operatorname{sgn} k\sqrt{\frac{|k|\pi}{\alpha}}, \operatorname{sgn} k\sqrt{\frac{|k+1|\pi}{\alpha}}]$, by Lemma 5.1. If $k \geqslant 1$ or $k \leqslant -2$, $\int_{T_k} |t - t_k| \cdot |t|^p\,\mathrm{d}t \leqslant \frac{1}{4} \cdot (\frac{3\pi}{\alpha})^{1+\frac{p}{2}}$. While for $k \in \{0, -1\}$, $\int_{T_k} |t - t_k| \cdot |t|^p\,\mathrm{d}t \leqslant (\frac{2\pi}{\alpha})^{1+\frac{p}{2}}$. Taking the larger of these yields $D_{\mathcal{T},p} \leqslant (\frac{2\pi}{\alpha})^{1+\frac{p}{2}}$ with $\mathcal{T} = \{t_k\}_{k\in\mathbb{Z}}$.

On the other hand, define $\lambda_k = \sqrt{\alpha\pi(4k - 1)}$ if $k \geqslant 1$ and $\lambda_k = -\sqrt{\alpha\pi(4|k - 1| - 1)}$ if $k \leqslant 0$. By Lemma 5.1, $\hat{f}(\lambda_k) = 0$ for each $k \in \mathbb{Z}$. Set $\Lambda_k = [\frac{1}{2}(\lambda_{k-1} + \lambda_k), \frac{1}{2}(\lambda_k + \lambda_{k+1})]$. We have $\int_{\Lambda_k} |\lambda - \lambda_k| \cdot \lambda^2\,\mathrm{d}\lambda \leqslant 16(\alpha\pi)^2$ for $k \geqslant 2$ or $k \leqslant -1$. On the other hand, if $k \in \{0, 1\}$, $\int_{\Lambda_k} |\lambda - \lambda_k| \cdot \lambda^2\,\mathrm{d}\lambda \leqslant 25(\alpha\pi)^2$. Taking the larger of these yields $D_{\Lambda,2} \leqslant 25(\alpha\pi)^2$ where $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$.

Finally, we conclude that $D_{\mathcal{T},p} D_{\Lambda,2} \leqslant 25\pi^2 (2\pi)^{1+\frac{p}{2}} \alpha^{1-\frac{p}{2}} < 1$. □

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