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A Note on compact hypersurfaces in a Euclidean space $\stackrel{\star}{\sim}$

Sur les hypersurfaces compactes d'un espace euclidien

Sharief Deshmukh

Differential Geometry

Department of Mathematics, College of Science, King Saud University, P.O. Box-2455, Riyadh 11451, Saudi Arabia

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ABSTRACT

In this Note, we show that the presence of a unit Killing vector field ξ on an orientable compact hypersurface of a Euclidean space with shape operator A and induced metric g such that $g(A\xi,\xi)$ is a constant, renders it to be a round sphere and also influences the dimension of the ambient Euclidean space.

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RÉSUMÉ

Supposons la donnée d'un champs de vecteurs unitaire de Killing ξ sur une hypersurface compacte, orientable, d'un espace euclidien, avec opérateur de forme *A* et métrique induite tels que $g(A\xi, \xi)$ soit constant. Nous montrons dans cette Note que cela impose que l'hypersurface est nécessairement isométrique à une sphère de courbure constante et que l'espace euclidien ambiant est de dimension paire.

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1. Introduction

Let (M, g) be an *n*-dimensional Riemannian manifold. There are several important types of smooth vector fields, whose existence influences the geometry of the Riemannian manifold (M, g). A smooth vector field ξ on *M* is said to be Killing if its local flow consists of local isometries of the Riemannian manifold (M, g). The presence of a nonzero Killing vector field on a compact Riemannian manifold constrains its geometry as well as topology, for instance, it does not allow the Riemannian manifold to have negative Ricci curvature and its fundamental group contains a cyclic subgroup with constant index depending only on *n* (cf. [1,2,9]). Also, it is known that on an even dimensional positive curved Riemannian manifold a Killing vector field must have a zero. The geometry of Riemannian manifolds with Killing vector fields has been studied quite extensively (cf. [1,2,8–10]). In this paper, we are interested in studying the impact of the presence of a unit Killing vector field on the geometry of a compact hypersurface of the Euclidean space. Recall that there are several characterizations of spheres among compact hypersurfaces in a Euclidean space and most of them involve some constraints on the curvatures of the hypersurfaces (cf. [4–7]). However, it is quite remarkable to note that mere presence of a unit Killing vector field ξ on a compact orientable hypersurface with shape operator *A* together with the condition that $g(A\xi, \xi)$ is a constant renders the hypersurface to be isometric to a round sphere (cf. main theorem). Our motivation comes through the geometry of the odddimensional sphere S^{2n-1} in the Euclidean space R^{2n} with unit normal vector field *N*, which admits a unit Killing vector field $\xi = -JN$ and $g(A\xi, \xi) = -1$, where *J* is the complex structure on the Euclidean space R^{2n} . This raises a question:

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"Does an orientable compact hypersurface in a Euclidean space that admits a unit Killing vector field ξ together with the condition that $g(A\xi, \xi)$ is a constant necessarily isometric to a sphere?". We answer this question in affirmative by proving the following:

Theorem. Let *M* be a compact orientable hypersurface of the Euclidean space \mathbb{R}^{n+1} . The hypersurface *M* admits a unit Killing vector field ξ with respect to the induced metric *g* and the shape operator *A* satisfying $g(A\xi, \xi)$ is a constant, if and only if n = 2m + 1 and *M* is isometric to the sphere $S^{2m+1}(c)$ of constant curvature *c*.

2. Preliminaries

Let (M, g) be a Riemannian manifold and ∇ be the Riemannian connection on it. A smooth vector field ξ on the Riemannian manifold (M, g) is said to be a Killing vector field if it satisfies

$$\mathcal{L}_{\xi}g = 0, \tag{1}$$

where $\mathcal{L}_{\xi}g$ is the Lie derivative of the metric g with respect to ξ . If η is smooth 1-form dual to the Killing vector field ξ on the Riemannian manifold (M, g), we define a skew-symmetric (1, 1) tensor field φ by

$$d\eta(X,Y) = 2g(\varphi X,Y), \quad X,Y \in \mathfrak{X}(M), \tag{2}$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on *M*. Then from Koszul's formula, together with Eqs. (1) and (2), give

$$\nabla_X \xi = \varphi X, \quad X \in \mathfrak{X}(M), \tag{3}$$

where ∇ is the Riemannian connection. Since the smooth 2-form $g(\varphi X, Y)$ is closed, we immediately get

$$(\nabla\varphi)(X,Y) = R(X,\xi)Y, \quad X,Y \in \mathfrak{X}(M), \tag{4}$$

where the covariant derivative $(\nabla \varphi)(X, Y) = \nabla_X(\varphi Y) - \varphi(\nabla_X Y)$ and *R* is the curvature tensor field of the Riemannian manifold (M, g).

Let *M* be an orientable hypersurface of the Euclidean space R^{n+1} with unit normal vector field *N* and the shape operator *A*. We denote the induced metric on *M* by *g* and the Riemannian connection with respect to the induced metric by ∇ . Then the Gauss and Codazzi equations for the hypersurface are

$$R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY, \quad X, Y, Z \in \mathfrak{X}(M)$$
(5)

and

$$(\nabla A)(X,Y) = (\nabla A)(Y,X), \quad X,Y \in \mathfrak{X}(M).$$
(6)

Suppose ξ is the unit Killing vector field on the hypersurface *M*. Then using skew-symmetry of the tensor field φ and Eq. (3), we get

$$\varphi(\xi) = 0 \quad \text{and} \quad \nabla_{\xi} \xi = 0. \tag{7}$$

As the vector field ξ is Killing, its flow $\{\psi_t\}$ consists of local isometries of M and thus we have $d\psi_t \circ A = A \circ d\psi_t$, which gives $\mathcal{L}_{\xi}A = 0$, that is

$$(\nabla A)(\xi, X) = \varphi A X - A \varphi X, \quad X \in \mathfrak{X}(M),$$

where we used Eq. (3). Using Eqs. (3) and (6) in the above equation, we get

$$\nabla_X A \xi = \varphi A X, \quad X \in \mathfrak{X}(M). \tag{8}$$

3. Proof of the theorem

Let *M* be a compact orientable hypersurface of the Euclidean space R^{n+1} and ξ be the unit Killing vector field on *M*. Suppose that $f = g(A\xi, \xi)$ is a constant. Then we set

$$A\xi = u + f\xi,\tag{9}$$

where the vector field $u \in \mathfrak{X}(M)$ is orthogonal to ξ . Taking covariant derivative with respect to $X \in \mathfrak{X}(M)$ in the above equation and using Eqs. (3) and (8), we arrive at

$$\nabla_X u = \varphi A X - f \varphi X. \tag{10}$$

Also, using X(f) = 0 and Eqs. (7), (8), (9), we get

$$0 = g(\nabla_X A\xi, \xi) + g(A\xi, \nabla_X \xi)$$
$$= g(u, \varphi X) = -g(\varphi u, X),$$

that is,

$$\varphi u = 0. \tag{11}$$

Using Eqs. (10) and (11), we conclude that $||u||^2$ is a constant function and as such if u vanishes at some point, then u = 0. Taking covariant derivative in Eq. (11) with respect to the vector field u and using Eq. (4), we get

$$R(u,\xi)u + \varphi(\nabla_u u) = 0,$$

which on taking the inner product with ξ gives the sectional curvature of the plane section spanned by $\{u, \xi\}$ as

 $R(u,\xi;\xi,u) = 0.$

However, on a compact hypersurface M of the Euclidean space there exists a point where all sectional curvatures are positive and hence at this point u must vanish, otherwise the above equation will give a contradiction. Hence u = 0 and we have

$$A\xi = f\xi.$$
(12)

Using Eqs. (8) and (12), we conclude that

$$\varphi(AX - fX) = 0, \quad X \in \mathfrak{X}(M). \tag{13}$$

Also, through Eqs. (3) and (4), we get that $R(X, \xi; \xi, X) = \|\varphi X\|^2$, $X \in \mathfrak{X}(M)$, and consequently, using Eqs. (5) and (12), we conclude that " $\varphi X = 0$ for X orthogonal to ξ if and only if fg(AX, X) = 0". Note that for any $X \in \mathfrak{X}(M)$, the vector field AX - fX is orthogonal to ξ and thus with the above argument, together with Eq. (13), give

$$fg(A(AX - fX), AX - fX) = 0, \quad X \in \mathfrak{X}(M).$$
(14)

Our next aim is to show that the constant f is a nonzero constant. To achieve this consider the position vector field Ψ of the hypersurface M in the Euclidean space \mathbb{R}^{n+1} , which we can express as $\Psi = t + \rho N$, where ρ is the support function and $t \in \mathfrak{X}(M)$. Then we have $\nabla_X t = X + \rho AX$, $X \in \mathfrak{X}(M)$ (cf. [2,7]). If we define a smooth function $h = g(t, \xi)$ on M, then its gradient is given by $\nabla h = (1 + f\rho)\xi - \varphi t$. At a critical point $q \in M$ of the function h we have $(1 + f\rho)(q)\xi_q = (\varphi t)(q)$, which on taking inner product with ξ_q , gives $1 + f\rho(q) = 0$ and this proves that the constant $f \neq 0$.

Finally suppose $AX = \lambda X$ in Eq. (14), we get

$$\lambda(\lambda - f)^2 = 0,$$

that is, there are two principal curvatures 0 and f and thus the hypersurface has two possible constant principal curvatures. Suppose that the hypersurface has exactly two constant principal curvatures 0, f with the eigen distributions

$$D_1 = \{X: AX = 0\}$$
 and $D_2 = \{X: AX = fX\},\$

which are orthogonal complementary smooth distributions. Moreover, using Eq. (6), it is easy to see that these distributions are involutive and parallel with leaves R^k and $S^{n-k}(c)$, $c = f^2$. However, by compactness of M, we get a contradiction. Hence there is only one constant principal curvature f and consequently A = fI, that is, M is a totally umbilical hypersurface and is therefore isometric to the sphere $S^n(c)$ of constant curvature $c = f^2$. Observe that n cannot be even, for otherwise the Killing vector field ξ has to vanish at some point, which is impossible as ξ is a unit vector field. Hence n = 2m + 1.

The converse is trivial.

4. Remarks

(i) We observe that the condition $g(A\xi,\xi)$ is a constant in the statement of the theorem is essential as there are examples of compact orientable hypersurfaces in a Euclidean space which admit unit Killing vector fields and are not isometric to a sphere. For example consider the smooth function $f: R^4 \rightarrow R$ defined by $f(x, y, z, w) = x^2 + y^2 + 4z^2 + 4w^2 - 1$ and the level set

$$M = f^{-1}\{0\},\$$

which is a compact orientable hypersurface of the Euclidean space R^4 . The vector field $\xi = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - 2w \frac{\partial}{\partial z} + 2z \frac{\partial}{\partial w}$ is a Killing vector field on the Euclidean space R^4 with global flow given by

)

$$\phi_t(x, y, z, w) = (x \cos t - y \sin t, x \sin t + y \cos t, z \cos 2t - w \sin 2t, z \sin 2t + w \cos 2t)$$

and the hypersurface M is invariant under this flow, consequently ξ is tangential to the hypersurface M and with respect to the induced metric g we have

$$g(\xi,\xi) = y^2 + x^2 + 4w^2 + 4z^2,$$

that is, ξ is a unit Killing vector field on the hypersurface M. The unit normal vector field N to the hypersurface is given by

$$N = \frac{1}{\sqrt{x^2 + y^2 + 16z^2 + 16w^2}} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 4z \frac{\partial}{\partial z} + 4w \frac{\partial}{\partial w} \right)$$

which together with the Euclidean connection D and the Euclidean metric \langle , \rangle on R^4 , gives

$$g(A\xi,\xi) = \langle N, D_{\xi}\xi \rangle = -\sqrt{x^2 + y^2 + 16z^2 + 16w^2},$$

which is not a constant on *M*.

(ii) A contact form η on a (2n + 1)-dimensional smooth manifold M is a smooth 1-form that satisfies $\eta \wedge (d\eta)^n \neq 0$ at each point of M. The pair (M, η) is called a contact manifold. On a contact manifold (M, η) , there exists a smooth vector field ξ called the Reeb vector field which satisfies $\eta(\xi) = 1$, $\xi \rfloor d\eta = 0$. Also, the contact manifold (M, η) admits a Riemannian metric g and a skew-symmetric (1, 1) tensor field φ satisfying

$$\begin{split} \eta(X) &= g(X,\xi), \qquad d\eta(X,Y) = g(X,\varphi Y), \qquad g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y), \\ \varphi(\xi) &= 0, \qquad \eta \circ \varphi = 0, \qquad \varphi^2 = -I + \eta \otimes \xi, \quad X,Y \in \mathfrak{X}(M), \end{split}$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. We call the structure (φ, ξ, η, g) a contact metric structure and the contact manifold with this structure is denoted by $M(\varphi, \xi, \eta, g)$ and we call it a contact metric manifold (cf. [3]). If in addition, the Reeb vector field ξ of the contact metric manifold is a Killing vector field, then $M(\varphi, \xi, \eta, g)$ is called a *K*-contact manifold. It is known that the sectional curvatures of the plane section containing the Reeb vector field are constant equal to 1. As an application of our theorem, we have the following:

Corollary. A (2n + 1)-dimensional compact K-contact manifold that admits an isometric immersion in the Euclidean space \mathbb{R}^{2n+2} with shape operator A such that $g(A\xi, \xi)$ is a constant is isometric to a sphere $S^{2n+1}(c)$.

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