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Number Theory

Special values of fractional hypergeometric functions for function fields

Valeurs spéciales des fonctions hypergéométriques fractionnaires pour les corps de fonctions

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ABSTRACT

In this work we study the fractional hypergeometric functions for function fields, introduced by D.S. Thakur. We shall characterize algebraic functions among them, and show the transcendence of special values at some nonzero algebraic arguments, in the case when they are entire transcendental functions.

RÉSUMÉ

Dans ce travail nous étudions les fonctions hypergéométriques fractionnaires pour les corps de fonctions, introduites par D.S. Thakur. Nous caractérisons celles de ces fonctions qui sont algébriques, et nous démontrons la transcendance de valeurs spéciales en certains arguments algébriques non nuls de celles qui sont des fonctions transcendantes entières.

Version française abrégée

Soit $p \geqslant 2$ un nombre premier et $q = p^{\theta}$ avec $\theta \geqslant 1$ un entier. Nous désignons par $\mathbb{F}_q$ le corps fini à q éléments, par $\mathbb{F}_q[T]$ l'anneau intègre des polynômes en T à coefficients dans $\mathbb{F}_q$, et par $\mathbb{F}_q(T)$ le corps de fractions de $\mathbb{F}_q[T]$. Pour tous les $P, Q \in \mathbb{F}_q[T]$ avec $Q \neq 0$, nous définissons $|P/Q|_\infty := q^{\deg P - \deg Q}$, et appelons $|\cdot|_\infty$ la valeur absolue ∞-adique sur $\mathbb{F}_q(T)$. Nous désignons par $\mathbb{F}_q((T^{-1}))$ le complété topologique de $\mathbb{F}_q(T)$ pour $|\cdot|_\infty$, et par $\mathbf{C}_\infty$ le complété topologique d'une clôture algébrique fixée de $\mathbb{F}_q((T^{-1}))$.

Pour tout $j \in \frac{1}{\theta}\mathbb{Z}$ ($j \geqslant 0$), posons

$$D_j = \prod_{k=0}^{\lceil j \rceil - 1} \left(T^{q^{j-k}} - T\right)^{q^k}, \quad \text{et} \quad L_j = \prod_{k=0}^{\lceil j \rceil - 1} \left(T^{q^{j-k}} - T\right),$$

où $\lceil j \rceil$ est la partie entière de j, i.e. le plus petit entier $\geqslant j$.

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Fixons $a \in \frac{1}{\theta}\mathbb{Z}$. Pour tous les entiers $n \geqslant 0$, définissons

$$(a)_n = \begin{cases} D_{n+a-1}^{q^{-(a-1)}} & \text{si } a > 0, \\ (-1)^{\lceil -a \rceil - n} L_{-a-n}^{-q^n} & \text{si } n \leqslant -a \text{ et } a \leqslant 0, \\ 0 & \text{si } n > -a \geqslant 0. \end{cases}$$

Suivant D.S. Thakur [8, p. 226], $\forall r, s \in \mathbb{N}$ et $\forall a_i, b_j \in \frac{1}{\theta}\mathbb{Z}$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s)$ avec $b_j > 0$ (de sorte qu'il n'existe pas de dénominateurs nuls), définissons la fonction hypergéométrique fractionnaire ${}_r\mathbf{F}_s$ comme :

$${}_r\mathbf{F}_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_r)_n}{D_n (b_1)_n \cdots (b_s)_n} z^{q^n},$$

et la désignons par ${}_r\mathbf{F}_s(z)$, quand les paramètres sont bien compris. Si $i \in \mathbb{N}$ $(1 \leqslant i \leqslant r)$ tel que $a_i \leqslant 0$, alors $(a_i)_n = 0$ pour tout $n > -a_i$. Dans ce cas ${}_r\mathbf{F}_s$ est un polynôme en z. Pour éviter cette trivialité, nous supposerons dans la suite $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$, $0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$, et $b_0 = 1$.

Par un calcul direct, nous obtenons tout de suite que le rayon de convergence R de ${}_r\mathbf{F}_s$ satisfait à

$$R = \begin{cases} 0 & \text{si } r > s+1, \\ q^{-\sum_{i=1}^{r}(\lceil a_i \rceil - 1) + \sum_{j=1}^{s}(\lceil b_j \rceil - 1)} & \text{si } r = s+1, \\ +\infty & \text{si } r < s+1. \end{cases}$$

Si $R = +\infty$, alors ${}_r\mathbf{F}_s$ est une fonction entière mais pas un polynôme selon notre hypothèse, elle est donc une fonction transcendante (voir par exemple Théorème 5 de [14]).

$\forall i, j \in \mathbb{Z}$ $(0 \leqslant i, j < \theta)$, posons $\delta(i, j) = 1$ si $i = j$, et 0 sinon. $\forall a \in \frac{1}{\theta}\mathbb{Z}$ $(a \geqslant 0)$, notons $\langle a \rangle := \theta \lceil a \rceil - \theta a$. $\forall i \in \mathbb{Z}$ $(0 \leqslant i < \theta)$, posons $a_i(0) = \sum_{t=1}^{r} \delta(i, \langle a_t \rangle)$, $b_i(0) = \sum_{t=0}^{s} \delta(i, \langle b_t \rangle)$, $c_i^-(0) = \max(0, b_i(0) - a_i(0))$.

Voici les résultats principaux :

Théorème 1. *Soient $r, s \geqslant 0$ deux entiers tels que $r = s + 1$, et $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$, $0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$ des rationnels dans $\frac{1}{\theta}\mathbb{Z}$. Alors ${}_r\mathbf{F}_s(z)$ est une fonction algébrique sur $\mathbb{F}_p(T, z)$ si et seulement si $({}_r\mathbf{F}_s(z))^{q^g} \in \mathbb{F}_p[T][[z]]$, avec $g = \max(\lceil a_r \rceil, \lceil b_s \rceil)$.*

Théorème 2. *Soient $r, s \geqslant 0$ deux entiers tels que $r < s+1$, et $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$, $0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$ des rationnels dans $\frac{1}{\theta}\mathbb{Z}$ tels que $q(s+1-r) > c := \sum_{i=0}^{\theta-1} c_i^-(0)$ (en particulier si $(1 - \frac{1}{q})(s+1) > r$). Alors ${}_r\mathbf{F}_s(\gamma)$ est transcendant, pour tout $\gamma \in \mathbf{C}_\infty \setminus \{0\}$ algébrique tel que $[\mathbb{F}_q(T, \gamma), \mathbb{F}_q(T)]_{\text{sep}} < q(s+1-r)/c$ ou que $\mathbb{F}_q(T, \gamma)$ possède une seule place sur la place à l'infini de $\mathbb{F}_q(T)$.*

1. Statements of main results

Classical hypergeometric functions are very important and have been studied by many authors. For example, F. Beukers and G. Heckman determined in [2] (following H.A. Schwarz in the simplest case) all the algebraic hypergeometric functions with rational parameters. For transcendental hypergeometric functions, there are many important results about the description, finiteness or infinitude of the set of algebraic arguments at which special values are also algebraic, for example, results by C.L. Siegel, A.B. Shidlovsky, J. Wolfart, P.B. Cohen, G. Wüstholz, W.D. Brownawell, F. Beukers. For more details and references, see for example the excellent surveys given by F. Beukers and P. Tretkoff at Arizona Winter School 2008, available as lecture notes at the website http://swc.math.arizona.edu/aws/2008/index.html.

D.S. Thakur introduced in 1995 two types of analogs ${}_rF_s$ and ${}_r\mathcal{F}_s$ of hypergeometric functions for function fields. See [8, 9], and [10, §6.5], [5] (see also [6,7]) for motivation and various properties such as analogues of Gauss differential equations they satisfy, good specializations, contiguous relations, transformations, connections with tensor powers of the Carlitz module, etc.

In this work we only discuss the first analog ${}_rF_s$, or rather its generalization from integral to fractional parameters (for the integral case, see [11] and [12]), which were suggested by D.S. Thakur in [8, p. 226] (but the detailed treatment was not published) for some fractional parameters when q is not a prime so that there are more roots of unity available than in the prime fields. Classically, special values of hypergeometric functions with fractional parameters are quite important. For example, they occur naturally in the study of periods and p-adic periods of elliptic curves (see [3,4], and also [1, §I.3]). It is conceivable, but not yet proved, that such a connection exists between the new function under consideration at half-integral parameters (when q is a square) and periods of rank two Drinfeld modules, parallel to the classical case, and functions with parameters having higher denominators connecting to higher rank, with no parallel to the classical case. Further, these fractional hypergeometric functions have interesting specializations studied by L. Carlitz.

Now we introduce some notation and definitions.

Fix $p \geqslant 2$ a prime number and $q = p^\theta$ with $\theta \geqslant 1$ an integer. Let $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{A} = \mathbb{F}_q[T]$ be the ring of polynomials in T over $\mathbb{F}_q$, and $\mathbb{K} = \mathbb{F}_q(T)$ be the fraction field of $\mathbb{A}$. For all $P, Q \in \mathbb{F}_q[T]$ and $Q \neq 0$, set

$|P/Q|_\infty := q^{\deg P - \deg Q}$. We denote by $\mathbb{F}_q((T^{-1}))$ the topological completion of $\mathbb{F}_q(T)$ with respect to $|\cdot|_\infty$, and by $\mathbb{L} = \mathbf{C}_\infty$ the topological completion of a fixed algebraic closure Ω of $\mathbb{F}_q((T^{-1}))$. Let $|\cdot|_\infty$ be the normalized absolute value on $\mathbf{C}_\infty$. For all $z \in \mathbf{C}_\infty$, set

$$\deg z := \log_q |z|_\infty := \frac{\log |z|_\infty}{\log q}, \quad \text{and} \quad v_\infty(z) := -\deg z.$$

Then $\deg z$ is just the usual degree of z if $z \in \mathbb{F}_q[T]$. We say that $\alpha \in \mathbf{C}_\infty$ is algebraic (resp. transcendental) if α is algebraic (resp. transcendental) over $\mathbb{F}_q(T)$.

For all rationals $j \in \frac{1}{\theta}\mathbb{Z}$ ($j \geqslant 0$), define

$$D_j = \prod_{k=0}^{\lceil j \rceil - 1} \left(T^{q^{j-k}} - T\right)^{q^k}, \quad \text{and} \quad L_j = \prod_{k=0}^{\lceil j \rceil - 1} \left(T^{q^{j-k}} - T\right),$$

where $\lceil j \rceil$ is the smallest integer $\geqslant j$. $\forall m \in \mathbb{N}$, put $[[m]] := T^{p^m} - T$. Then for all integers $n \geqslant 0$,

$$D_{\frac{n}{\theta}} = \prod_{k=0}^{\lceil \frac{n}{\theta} \rceil - 1} [[n - \theta k]]^{q^k}, \quad \text{and} \quad L_{\frac{n}{\theta}} = \prod_{k=0}^{\lceil \frac{n}{\theta} \rceil - 1} [[n - \theta k]].$$

Fix $a \in \frac{1}{\theta}\mathbb{Z}$. For all integers $n \geqslant 0$, define

$$(a)_n = \begin{cases} D_{n+a-1}^{q^{-(a-1)}} & \text{if } a > 0, \\ (-1)^{\lceil -a \rceil - n} L_{-a-n}^{-q^n} & \text{if } n \leqslant -a \text{ and } a \leqslant 0, \\ 0 & \text{if } n > -a \geqslant 0. \end{cases}$$

Following D.S. Thakur [8, p. 226], for all integers $r, s \geqslant 0$ and for all $a_i, b_j \in \frac{1}{\theta}\mathbb{Z}$ ($1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$) with $b_j > 0$ (to avoid zero denominators), we define the fractional hypergeometric function ${}_r\mathbf{F}_s$ as follows:

$${}_r\mathbf{F}_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_r)_n}{D_n (b_1)_n \cdots (b_s)_n} z^{q^n},$$

and denote it by ${}_r\mathbf{F}_s(z)$, when the parameters are well understood. If there exists some integer i ($1 \leqslant i \leqslant r$) such that $a_i \leqslant 0$, then for $n > -a_i$, we have $(a_i)_n = 0$. In this case ${}_r\mathbf{F}_s$ is a polynomial in z. To avoid triviality, in the following we shall always suppose $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r$, and $0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$, then the radius of convergence R of ${}_r\mathbf{F}_s$ satisfies

$$R = \begin{cases} 0 & \text{if } r > s + 1, \\ q^{-\sum_{i=1}^r (\lceil a_i \rceil - 1) + \sum_{j=1}^s (\lceil b_j \rceil - 1)} & \text{if } r = s + 1, \\ +\infty & \text{if } r < s + 1. \end{cases}$$

If $R = +\infty$, then ${}_r\mathbf{F}_s$ is entire but not a polynomial, so it is transcendental (see [14, Theorem 5]).

Set $g = \max(\lceil a_r \rceil, \lceil b_s \rceil)$. For all integers i, j ($0 \leqslant i, j < \theta$), define $\delta(i, j) = 1$ if $i = j$, and 0 otherwise. For all $a \in \frac{1}{\theta}\mathbb{Z}$ ($a \geqslant 0$), put $\langle a \rangle := \theta \lceil a \rceil - \theta a$. Then for all $i, j \in \mathbb{Z}$ ($0 \leqslant i < \theta$), set

$$a_i(j) = \sum_{t=k}^{r} \delta\big(i, \langle a_t \rangle\big) \quad \text{if } \lceil a_{k-1} \rceil \leqslant j \leqslant \lceil a_k \rceil - 1,$$

$$b_i(j) = \sum_{t=k}^{s} \delta\big(i, \langle b_t \rangle\big) \quad \text{if } \lceil b_{k-1} \rceil \leqslant j \leqslant \lceil b_k \rceil - 1,$$

$$c_i(j) = a_i(j) - b_i(j), \qquad c_i^-(j) = \max\big(0, -c_i(j)\big) \leqslant s + 1,$$

where by convention, $b_0 = 1$, $a_0 = b_{-1} = -\infty$, and $a_{r+1} = b_{s+1} = +\infty$. Then $c_i(j) = 0$ if $j \geqslant g$.

The main results are the following:

Theorem 1. *Let $r, s \geqslant 0$ be integers such that $r = s + 1$, and let*

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \quad \textit{and} \quad 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

be rationals in $\frac{1}{\theta}\mathbb{Z}$. Then the following properties are equivalent:

(i) *For all* $i, j \in \mathbb{Z}$ $(0 \leqslant i < \theta)$, *we have* $c_i(j) \geqslant 0$;
(ii) $({}_r\mathbf{F}_s(z))^{q^g} \in \mathbb{F}_p[T][[z]]$;
(iii) ${}_r\mathbf{F}_s(z)$ *is an algebraic function over* $\mathbb{F}_p(T, z)$.

Theorem 2. *Let* $r, s \geqslant 0$ *be integers such that* $r < s + 1$, *and let*

$$0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \quad \text{and} \quad 0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$$

be rationals in $\frac{1}{\theta}\mathbb{Z}$ *such that* $q(s+1-r) > c := \sum_{i=0}^{\theta-1} c_i^-(0)$ (*in particular if* $(1-\frac{1}{q})(s+1) > r$). *Then* ${}_r\mathbf{F}_s(\gamma)$ *is transcendental, for all algebraic* $\gamma \in \mathbf{C}_\infty \backslash \{0\}$ *such that* $[\mathbb{F}_q(T,\gamma), \mathbb{F}_q(T)]_{\mathrm{sep}} < q(s+1-r)/c$ *or* $\mathbb{F}_q(T,\gamma)$ *has only one place over the usual infinite place of* $\mathbb{F}_q(T)$.

The first generalizes Theorem 4 in [12] (see also [11, Theorem 3]), and gives an exact analog of the result of F.R. Villegas [13], based on a result of F. Beukers and G. Heckman [2] in the classical hypergeometric case; the second corrects, improves, and generalizes Theorem 2 in [12] (see also [11, Theorem 1]). More precisely, the hypothesis that "$\mathbb{F}_q(T,\gamma)$ has less than q places above the infinite place of $\mathbb{F}_q[T]$" in Theorem 2 in [12] (see also [11, Theorem 1]) (repeated in the abstract and the introduction) should be replaced by "the degree D of $\mathbb{F}_q(T,\gamma)$ over $\mathbb{F}_q(T)$ is less than q". The corresponding changes in the proof in [12] are replacing "$v^{(i)}$" by "$n_i v^{(i)}$" in the 6-th displayed formula of page 146 and "d" by "D" in the 7-th displayed formula (here n_i are local degrees of $v^{(i)}$ normalized to extend the valuation at base).

2. Proofs (sketches)

For all integers $n \geqslant 0$, set $u(n) = \prod_{i=0}^{\theta-1} \prod_{j=1}^{g+n-1} [[\theta j - i]]^{c_i(j-n)p^{\theta(g+n-j)+i}}$.

Proof of Theorem 1. (1) $\Rightarrow$ (2) comes directly from the formula $({}_r\mathbf{F}_s(z))^{q^g} = \sum_{n=0}^{+\infty} u(n) z^{q^{g+n}}$.

(2) $\Rightarrow$ (1): Fix $i_0, j_0 \in \mathbb{Z}$ $(0 \leqslant i_0 < \theta)$. If $j_0 \geqslant g$, then $a_{i_0}(j_0) = b_{i_0}(j_0) = 0$, and thus $c_{i_0}(j_0) = 0$. So we need only consider $j_0 < g$. Let $f_n \in \mathbb{F}_p[T]$ be monic and irreducible of degree $n \geqslant 1$. $\forall P \in \mathbb{F}_p[T]$, put $\nu_n(P) = \sup\{k \in \mathbb{N}\colon\ f_n^k \mid P\}$. Then extend the valuation ν_n over $\mathbb{F}_p(T)$, and in turn over $\overline{\mathbb{F}_p(T)}$. $\forall m \in \mathbb{N}$, we have that f_n divides $[[m]]$ in $\mathbb{F}_p[T]$ if and only if n divides m, and in the latter case $\nu_n([[m]]) = 1$. Now $({}_rF_s(z))^{q^g} \in \mathbb{F}_p[T][[z]]$, so $0 \leqslant \nu_{\theta(n+j_0)-i_0}(u(n)) = c_{i_0}(j_0)p^{\theta(g-j_0)+i_0}$ for all integers $n > g + 2|j_0| + 1$. By the way, we obtain $c_i(0) = 0$ $(0 \leqslant i < \theta)$ since $c_i(0) \geqslant 0$ $(0 \leqslant i < \theta)$ and $\sum_{i=0}^{\theta-1} c_i(0) = r - (s+1) = 0$.

(2) $\Rightarrow$ (3): Fix $i \in \mathbb{Z}$ $(0 \leqslant i < \theta)$. For all integers n, k $(n \geqslant 0,\ 1 \leqslant k \leqslant n)$, we have $c_i(k-n) = c_i(0) = 0$. So the coefficient of $z^{q^{g+n}}$ in $({}_rF_s(z))^{q^g}$ equals $\prod_{i=0}^{\theta-1} \prod_{k=1}^{g-1} (T^{q^{g+n}} - T^{p^{\theta(g-k)+i}})^{c_i(k)}$, which is a finite sum of $\pm T^{mq^{g+n}+h}$ with $m, h \geqslant 0$ bounded integers. Then $({}_r\mathbf{F}_s(z))^{q^g}$ is a finite $\mathbb{F}_p[T]$-linear combination of the formal power series of the form $G_m(z) = \sum_{n=0}^{+\infty} T^{mq^{g+n}} z^{q^{g+n}}$. However we have

$$\big(G_m(z)\big)^q = \sum_{n=0}^{+\infty} T^{mq^{g+n+1}} z^{q^{g+n+1}} = \sum_{n=1}^{+\infty} T^{mq^{g+n}} z^{q^{g+n}} = G_m(z) - T^{mq^g} z^{q^g},$$

hence $G_m(z)$ is an algebraic function, and so is ${}_r\mathbf{F}_s(z)$.

(3) $\Rightarrow$ (1): Since ${}_r\mathbf{F}_s(z)$ is an algebraic function, then by Theorem 4 in [16], the $\overline{\mathbb{F}_p(T)}$-vector space generated by the sequences $(u^{1/q^k}(n+k))_{n \geqslant 0}$ $(k \in \mathbb{N})$ has finite dimension, thus we can find an integer $t \geqslant 1$ such that for all integers $k \geqslant 0$, $\exists A_{k,m} \in \overline{\mathbb{F}_p(T)}$ $(0 \leqslant m \leqslant t)$ such that

$$u^{1/q^k}(n+k) = \sum_{m=0}^{t} A_{k,m} u^{1/q^m}(n+m) \quad (\forall n \geqslant 0).$$

By contradiction, assume that $\exists i_0, j_0 \in \mathbb{Z}$ satisfying $0 \leqslant i_0 < \theta$, $j_0 < g$, and $c_{i_0}(j_0) < 0$. For all integers $n \geqslant 1$, let ν_n be the valuation over $\overline{\mathbb{F}_p(T)}$ defined above. Fix $k > t + g + 2|j_0|$. As above, $\forall n \in \mathbb{N}$, $\nu_{\theta(n+k+j_0)-i_0}(u(n+k)) = c_{i_0}(j_0)p^{\theta(g-j_0)+i_0}$, and $\nu_{\theta(n+k+j_0)-i_0}(u(n+m)) = 0$ $(0 \leqslant m \leqslant t)$. For all integers m $(0 \leqslant m \leqslant t)$, if $A_{k,m} \neq 0$, then $\nu_{\theta(n+k+j_0)-i_0}(A_{k,m}) = 0$ for all integers n large enough, so

$$c_{i_0}(j_0)p^{\theta(g-j_0-k)+i_0} = \nu_{\theta(n+k+j_0)-i_0}\big(u^{1/q^k}(n+k)\big) \geqslant \min_{0 \leqslant m \leqslant t} \nu_{\theta(n+k+j_0)-i_0}\big(A_{k,m} u^{1/q^m}(n+m)\big) = 0.$$

Absurd. So we must have $({}_r\mathbf{F}_s(z))^{q^g} \in \mathbb{F}_p[T][[z]]$. $\square$

Proof of Theorem 2. Let $\gamma \in \mathbf{C}_\infty \backslash \{0\}$ be algebraic over $\mathbb{F}_q(T)$ with separability degree χ. Let $\ell \geqslant 1$ be an integer such that γ^{q^ℓ} is separable over $\mathbb{F}_q(T)$. Set $\mathbf{K} = \mathbb{F}_q(T, \gamma^{q^\ell})$. Denote by $\mathcal{O}_{\mathbf{K}}$ the integral closure of $\mathbb{F}_q[T]$ in $\mathbf{K}$. Put $\alpha := ({}_r\mathbf{F}_s(\gamma))^{q^{g+\ell}} = \sum_{n=0}^{+\infty} u(n)^{q^\ell} \gamma^{q^{g+\ell+n}}$. For all integers $m \geqslant 0$, define $w_m = u(m)^{q^\ell} \gamma^{q^{g+\ell+m}}$, $\alpha_m = \sum_{n=0}^{m} w_n$, and $\delta_m = |\alpha - \alpha_m|_\infty$.

Fix $t \geqslant 1$ an integer, and $A_0, A_1, \ldots, A_t \in \mathbb{F}_q[T]$ not all zero. We denote by ρ the least integer such that $A_\rho \neq 0$. To simplify the notation, we suppose without loss of generality $\rho = 0$. The general case can be proved similarly and directly but with much more complicated notation.

$\forall m \in \mathbb{N}$ $(m \geqslant t)$, set $\beta_m = \sum_{j=0}^{t} A_j \alpha_{m-j}^{q^j}$. Since $\sum_{i=0}^{\theta-1} c_i(0) = r-(s+1) < 0$, thus $\exists i_0 \in \mathbb{Z}$ $(0 \leqslant i_0 < \theta)$ such that $c_{i_0}(0) < 0$. But $c_{i_0}(j) = 0$ for all integers $j \geqslant g$, so we can find a greatest integer j_0 such that $0 \leqslant j_0 < g$, $c_{i_0}(j_0) < 0$, and $c_{i_0}(j) \geqslant 0$ for all integers $j > j_0$. For all integers $n \geqslant 1$, let $f_n \in \mathbb{F}_q[T]$ be monic and irreducible of degree n. As above, it can induce a valuation ν_n over $\mathbb{F}_q(T)$, which we extend over $\mathbf{K}$. Choose $N \geqslant t+g+\ell+\max_{0\leqslant j\leqslant t} \deg A_j$ such that for all integers $n > N$, $\nu_n(\gamma) = 0$. Fix $m > N$ an integer. For all integers n $(0 \leqslant n \leqslant m)$, from the definition of w_n and the fact that $m > g$, and $c_i(j-n) = 0$ if $j-n \geqslant g$, one checks at once

$$\nu_{\theta(j_0+m)-i_0}(w_n) = c_{i_0}(j_0+m-n)p^{\theta(g+\ell+n-j_0-m)+i_0}.$$

If $0 \leqslant n < m$, then $j_0+m-n > j_0$, and $c_{i_0}(j_0+m-n) \geqslant 0$. So for all integers k $(0 \leqslant k < m)$, we have $\nu_{\theta(j_0+m)-i_0}(\alpha_k) \geqslant \min_{0\leqslant n\leqslant k} \nu_{\theta(j_0+m)-i_0}(w_n) \geqslant 0$. Now for all integers n $(0 \leqslant n < m)$, we have

$$\nu_{\theta(j_0+m)-i_0}(w_n) \geqslant 0 > c_{i_0}(j_0)p^{\theta(g+\ell-j_0)+i_0} = \nu_{\theta(j_0+m)-i_0}(w_m),$$

hence $\nu_{\theta(j_0+m)-i_0}(\alpha_m) = \nu_{\theta(j_0+m)-i_0}(\sum_{n=0}^{m} w_n) = c_{i_0}(j_0)p^{\theta(g+\ell-j_0)+i_0} < 0$. Consequently we obtain

$$\nu_{\theta(j_0+m)-i_0}(\beta_m) = \nu_{\theta(j_0+m)-i_0}\left(\sum_{j=0}^{t} A_j \alpha_{m-j}^{q^j}\right) = \nu_{\theta(j_0+m)-i_0}(A_0\alpha_m) = \nu_{\theta(j_0+m)-i_0}(\alpha_m) < 0.$$

So $\beta_m \neq 0$. Let $E \in \mathbb{F}_q[T]\backslash\{0\}$ be such that $E\gamma \in \mathcal{O}_{\mathbf{K}}$. Set

$$H_m = E^{q^{g+\ell+m}} \prod_{i=0}^{\theta-1}\left(\prod_{j=1}^{m} [[\theta j - i]]^{c_i^-(0)p^{\theta(g+\ell+m-j)+i}} \cdot \prod_{j=1}^{g+m-1} [[\theta j - i]]^{(s+1)p^{\theta(g+\ell+t)+i}}\right).$$

Then $\deg H_m \leqslant q^{g+\ell+m}(\deg E + cm + (s+1)q^{g+t})$. Note that for all integers $m > N$, we have

$$\beta_m = \sum_{k=0}^{t} A_k \sum_{n=0}^{m-k} \gamma^{q^{k+g+\ell+n}} \cdot \prod_{i=0}^{\theta-1}\left(\prod_{j=1}^{n} [[\theta j - i]]^{c_i(0)p^{\theta(k+g+\ell+n-j)+i}} \cdot \prod_{j=1}^{g-1} \big[\big[\theta(n+j) - i\big]\big]^{c_i(j)p^{\theta(k+g+\ell-j)+i}}\right).$$

So $H_m\beta_m \in \mathcal{O}_{\mathbf{K}} \setminus \{0\}$. By Theorem 2-4-6 and Proposition 2-5-1 in [15], we can find integers $\chi_i \geqslant 1$ and valuations $\nu^{(i)}$ $(1 \leqslant i \leqslant d)$ over $\mathbf{K}$ which extend the usual ∞-adic valuation over $\mathbb{F}_q(T)$ such that

$$\sum_{i=1}^{d} \chi_i = \chi, \quad \text{and} \quad \sum_{i=1}^{d} \chi_i \nu^{(i)}(x) = \nu_\infty\big(\mathrm{N}(x)\big), \quad \forall x \in \mathbf{K},$$

where $\mathrm{N}(x)$ is the norm of x for the field extension $\mathbf{K}/\mathbb{F}_q(T)$, so $\mathrm{N}(H_m\beta_m) \in \mathbb{F}_q[T] \setminus \{0\}$. But $\nu^{(i)}(\beta_m)$ $(1 \leqslant i \leqslant d)$ is bounded below independently of m, thus $\exists C > 0$ such that for sufficiently large integers m,

$$\nu_\infty(\beta_m) = -\nu_\infty\big(\mathrm{N}(H_m)\big) + \nu_\infty\big(\mathrm{N}(H_m\beta_m)\big) + \left(\nu_\infty(\beta_m) - \sum_{i=1}^{d} \chi_i \nu^{(i)}(\beta_m)\right) \leqslant \chi \deg H_m + C.$$

Note that for all sufficiently large integers m, we have

$$\delta_m \leqslant \sup_{n\geqslant m+1} |w_n| \leqslant q^{q^{g+\ell+m+1}((r-s-1)(m+1)+\sum_{i=1}^{r}(\lceil a_i\rceil-1)-\sum_{j=1}^{s}(\lceil b_j\rceil-1)+\deg\gamma)},$$

hence for all integers j $(0 \leqslant j \leqslant t)$, we have, for $m \to +\infty$,

$$\nu_\infty(\beta_m) + q^j \log_q \delta_{m-j} \precsim -q^{g+\ell+m}\big(q(s+1-r) - \chi c\big)m \to -\infty.$$

So $({}_r\mathbf{F}_s(\gamma))^{q^{g+\ell}}$ (and thus ${}_r\mathbf{F}_s(\gamma)$) is transcendental by virtue of Theorem 1 in [12].

Moreover if $d = 1$, then for all integers $m \geqslant 0$, we have $\chi\nu_\infty(H_m\beta_m) = \nu_\infty(\mathrm{N}(H_m\beta_m)) \leqslant 0$, so $\nu_\infty(\beta_m) \leqslant -\nu_\infty(H_m) = \deg H_m$. Then for all integers j $(0 \leqslant j \leqslant t)$, we have, for $m \to +\infty$,

$$\nu_\infty(\beta_m) + q^j \log_q \delta_{m-j} \precsim -q^{g+\ell+m}\big(q(s+1-r) - c\big)m \to -\infty.$$

So ${}_r\mathbf{F}_s(\gamma)$ is transcendental again as above by virtue of Theorem 1 in [12]. □

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