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Numerical Analysis

A robust variant of NXFEM for the interface problem

Une variante robuste de NXFEM pour le problème d'interface

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A R T I C L E   I N F O

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A B S T R A C T

In this note, we propose a modification of the NXFEM proposed in Hansbo and Hansbo (2002) [4] for the elliptic interface problem. It leads to a robust method not only with respect to the mesh-interface geometry, but also with respect to the diffusion parameters.

R É S U M É

Dans cette note, nous proposons une modification de NXFEM proposée dans Hansbo et Hansbo (2002) [4] pour le problème d'interface elliptique. Elle permet d'obtenir la robuste à la fois par rapport à la géometrie du maillage coupé par l'interface et par rapport aux paramètres de diffusion.

1. Introduction

The Nitsche extended finite element method (NXFEM) has been proposed in [4] for the interface problem, formulated for simplicity with a bounded polygonal domain Ω, homogenous Dirichlet conditions, and $f \in L^2(\Omega)$,

$$\operatorname{div}(\sigma) = f, \qquad \sigma = -k\nabla u, \qquad u|_{\partial\Omega} = 0, \tag{1}$$

where k is a discontinuous coefficient taking values k_{in} and $k_{\text{ex}}\,(\neq k_{\text{in}})$ on two disjoint subdomains Ω_{in} and Ω_{ex}, $\Omega = \Omega_{\text{in}} \cup \Omega_{\text{ex}}$. Although the mesh is not supposed to be aligned with the discontinuity of k, the method has excellent accuracy, since it employs an appropriate finite element space for approximation of the non-regular solution of (1). This finite element space is obtained by 'doubling' the local function space on all triangles cut by the interface $\Gamma := \overline{\Omega_{\text{in}}} \cap \overline{\Omega_{\text{ex}}}$, which is here assumed to be polygonal. The discontinuity of the finite element space is then handled by a variant of Nitsche's method, making use of the interface conditions

$$[u] = g_D \quad \text{and} \quad [\sigma \cdot n] = g_N \quad \text{on } \Gamma, \tag{2}$$

where $g_D \in C(\Gamma)$ and $g_N \in L^2(\Gamma)$ are given functions, and n is the unit outer normal field of Ω_{in}.

In [4] the stability of the method and optimal order error estimates are proven, supposing only that the solution is piecewise in H^2. The method is robust with respect to the geometry of the cut triangles. However, the original NXFEM is not robust with respect to the coefficients k_{in}, k_{ex}.

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In this note, we propose a modification of the Hansbo–Hansbo method, which is not only robust with respect to the cut geometry but also with respect to the coefficients. This allows us in particular to use it as a fictitious domain method.

Other robust formulations of NXFEM are available, either employing additional meshes [5,1], or introducing stabilization [2]. The method proposed here has some similarities with the stabilized approach, but avoids any additional terms.

2. The original Hansbo–Hansbo method

We denote by $\mathcal{H}$ a family of uniformly shape-regular meshes h consisting of triangles $\mathcal{K}_h$. We define the set of cut cells by $\mathcal{K}_h^{\text{in}} := \{K \cap \Omega_{\text{in}}\colon\ K \in \mathcal{K}_h \text{ such that } K \cap \Omega_{\text{in}} \neq \emptyset\}$ and similarly for $\mathcal{K}_h^{\text{ex}}$. We then define the finite element space, with $C_0(A) := \{u \in C(\overline{A})\colon\ u|_{\partial\Omega\cap\partial A} = 0\}$,

$$V_h := \left\{v_h \in L^2(\Omega),\ v_h|_{\Omega_{\text{in}}} \in C_0(\Omega_{\text{in}}),\ v_h|_{\Omega_{\text{ex}}} \in C_0(\Omega_{\text{ex}}),\ v_h|_M \in P^1(M)\ \forall M \in \mathcal{K}_h^{\text{in}} \cup \mathcal{K}_h^{\text{ex}}\right\}. \tag{3}$$

The functions of V_h are discontinuous across the interface which is divided into the set of segments

$$\mathcal{S}_h^{\Gamma} := \{K \cap \Gamma\colon\ K \in \mathcal{K}_h \text{ such that } K \cap \Gamma \neq \emptyset\}.$$

Let $v_h \in V_h$. For a given side $S \in \mathcal{S}_h^{\Gamma}$ we fix a unit normal n_S once for all and define for $x \in S$

$$v_{hS}^{\text{in}}(x) := \lim_{\varepsilon \searrow 0} v_h(x - \varepsilon n_S), \qquad v_{hS}^{\text{ex}}(x) = \lim_{\varepsilon \searrow 0} v_h(x + \varepsilon n_S).$$

Next we define the jump and weighted mean for a weight $\alpha = (\alpha^{\text{in}}, \alpha^{\text{ex}})$ with $\alpha^{\text{in}} + \alpha^{\text{ex}} = 1$ by

$$[u](x) := u_S^{\text{in}}(x) - u_S^{\text{ex}}(x), \qquad \{u\}_\alpha(x) := \alpha^{\text{in}} u_S^{\text{in}}(x) + \alpha^{\text{ex}} u_S^{\text{ex}}(x).$$

The following formula, where $\hat{\alpha} = (\alpha^{\text{ex}}, \alpha^{\text{in}})$ readily follows from these definitions:

$$[uv] = [u]\{v\}_\alpha + \{u\}_{\hat{\alpha}}[v]. \tag{4}$$

We define the linear functional

$$l(v_h) := \int_\Omega f v_h + \int_{\mathcal{S}_h^\Gamma} g_N \{v_h\}_{\hat{\alpha}} - \int_{\mathcal{S}_h^\Gamma} g_D\, \partial_{n,k}^* v_h, \tag{5}$$

and the symmetric bilinear form

$$a_h(u_h, v_h) := \sum_{M \in \mathcal{K}_h^{\text{in}} \cup \mathcal{K}_h^{\text{ex}}} \int_M k\nabla u_h \cdot \nabla v_h - \sum_{S \in \mathcal{S}_h^\Gamma} \int_S \left([u_h]\partial_{n,k}^* v_h + \{\partial_{n,k} u_h\}_{\alpha_S}[v_h]\right), \tag{6}$$

where we have used the discrete fluxes $\partial_{n,k}^* v_h|_S := \{\partial_{n,k} v_h\}_{\alpha_S} - \gamma_S[v_h]$, $\partial_{n,k} v_h := n_S^T k \nabla v_h$, which depend on the two parameters α_S and γ_S, constant on each segment S. Their precise choice is crucial for stability and robustness of the method. The method is consistent for any choice of parameters.

Lemma 2.1. *Let u be a smooth solution to (1), (2). Then for any $v_h \in V_h$, we have $a_h(u, v_h) = l(v_h)$.*

Proof. By the integration by parts formula

$$\sum_{M \in \mathcal{K}_h^{\text{in}} \cup \mathcal{K}_h^{\text{ex}}} \int_M k\nabla u \cdot \nabla v_h = -\int_\Omega \operatorname{div}(k\nabla u) v_h + \sum_{S \in \mathcal{S}_h^\Gamma} \int_S [\partial_{n,k} u v_h],$$

(4), and the second interface condition (2) we have

$$[\partial_{n,k} u v_h] = \{\partial_{n,k} u\}_\alpha [v_h] + [\partial_{n,k} u]\{v_h\}_{\hat{\alpha}} = \{\partial_{n,k} u\}_\alpha [v_h] + g_N \{v_h\}_{\hat{\alpha}}.$$

Together with the first interface condition (2) this yields

$$a_h(u, v_h) = \int_\Omega f v_h + \int_{\mathcal{S}_h^\Gamma} g_N \{v_h\}_{\hat{\alpha}} - \int_{\mathcal{S}_h^\Gamma} g_D\, \partial_{n,k}^* v_h = l(v_h). \qquad \square$$

The choice of the parameters α_S and γ_S is therefore guided by the stability analysis. We denote by $\|\cdot\|_A$ the $L^2(A)$-norm (suppressing the subscript in case $A : \Omega$) and by ∇_h the piecewise gradient operator. We introduce $\|v_h\|_{h,\Gamma}^2 := \sum_{S \in \mathcal{S}_h^\Gamma} \gamma_S \|[v_h]\|^2$ and the norm $|||v_h|||_h^2 := \|k\nabla_h v_h\|^2 + \|v_h\|_{h,\Gamma}^2$. We note that $a_h(u_h, u_h) = |||u_h|||_h^2 - 2\int_S [u_h]\{\partial_{n,k} u_h\}_{\alpha_S}$.

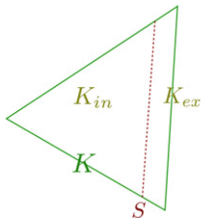

Fig. 1. Cut triangle.

The last term is bounded with the help of Young's inequality as

$$\int_S [u_h]\{\partial_{n,k}u_h\}_{\alpha_S} \leqslant \frac{1}{4}\|u_h\|^2_{h,\Gamma} + \sum_{S\in\mathcal{S}_h^\Gamma} \gamma_S^{-1}\big\|\{\partial_{n,k}u_h\}_{\alpha_S}\big\|^2_S.$$

Therefore, the coercivity of a_h depends on an estimate of the type $\sum_{S\in\mathcal{S}_h^\Gamma}\gamma_S^{-1}\|\{\partial_{n,k}u_h\}_{\alpha_S}\|^2_S \leqslant \frac{1}{4}\|k\nabla u_h\|^2$. Let $S\in\mathcal{S}_h^\Gamma$ be a side cutting the element $K=K^{\mathrm{in}}\cup K^{\mathrm{ex}}$ (see Fig. 1). The choice of parameters in the Hansbo–Hansbo method is with a mesh- and geometry-independent constant $\hat{\gamma}>0$

$$\alpha_S^{\mathrm{in}} = \frac{|K^{\mathrm{in}}|}{|K|}, \qquad \alpha_S^{\mathrm{ex}} = \frac{|K^{\mathrm{ex}}|}{|K|}, \qquad \gamma_S := \hat{\gamma}\max\left\{k_{\mathrm{in}}\frac{|K^{\mathrm{in}}|}{|K|}, k_{\mathrm{ex}}\frac{|K^{\mathrm{ex}}|}{|K|}\right\}\frac{|S|}{|K|}. \tag{7}$$

The stability, independent of the cut geometry, is shown as follows. Since the gradient of u_h is constant on each cut element, we have for example for K^{in}

$$\int_S |\partial_n u_h|^2_{K^{\mathrm{in}}} \leqslant \frac{|S|}{|K^{\mathrm{in}}|}\int_{K^{\mathrm{in}}} |\nabla u_h|^2. \tag{8}$$

Using that ∇u_h is constant on each cell, it follows that

$$\frac{1}{2}\int_S \{\partial_{n,k}u_h\}^2_{\alpha_S} \leqslant \left(\alpha_S^{\mathrm{in}}\right)^2\int_S k_{\mathrm{in}}^2|\partial_n u_h|^2_{K^{\mathrm{in}}} + \left(\alpha_S^{\mathrm{ex}}\right)^2\int_S k_{\mathrm{ex}}^2|\partial_n u_h|^2_{K^{\mathrm{ex}}} \leqslant \left(\alpha_S^{\mathrm{in}}\right)^2\frac{k_{\mathrm{in}}|S|}{|K^{\mathrm{in}}|}\int_{K^{\mathrm{in}}} k_{\mathrm{in}}|\nabla u_h|^2$$
$$+ \left(\alpha_S^{\mathrm{ex}}\right)^2\frac{k_{\mathrm{ex}}|S|}{|K^{\mathrm{ex}}|}\int_{K^{\mathrm{ex}}} k_{\mathrm{ex}}|\nabla u_h|^2 \leqslant \frac{k_{\mathrm{in}}|S||K^{\mathrm{in}}|}{|K|^2}\int_{K^{\mathrm{in}}} k_{\mathrm{in}}|\nabla u_h|^2 + \frac{k_{\mathrm{ex}}|S||K^{\mathrm{ex}}|}{|K|^2}\int_{K^{\mathrm{ex}}} k_{\mathrm{ex}}|\nabla u_h|^2 \leqslant \frac{\gamma_S}{\hat{\gamma}}\int_K k|\nabla u_h|^2.$$

This implies the stability of the method with respect to the cut geometry. Optimal order error estimates are derived in [4].

3. A robust method

Let $S\in\mathcal{S}_h^\Gamma$ be a side cutting the element $K=K^{\mathrm{in}}\cup K^{\mathrm{ex}}$. We define with a mesh- and geometry-independent constant $\hat{\gamma}>0$

$$\alpha_S^{\mathrm{in}} = \frac{k_{\mathrm{ex}}|K^{\mathrm{in}}|}{k_{\mathrm{ex}}|K^{\mathrm{in}}|+k_{\mathrm{in}}|K^{\mathrm{ex}}|}, \qquad \alpha_S^{\mathrm{ex}} = \frac{k_{\mathrm{in}}|K^{\mathrm{ex}}|}{k_{\mathrm{ex}}|K^{\mathrm{in}}|+k_{\mathrm{in}}|K^{\mathrm{ex}}|}, \qquad \gamma_S := \hat{\gamma}\frac{k_{\mathrm{in}}k_{\mathrm{ex}}|S|}{k_{\mathrm{ex}}|K^{\mathrm{in}}|+k_{\mathrm{in}}|K^{\mathrm{ex}}|}. \tag{9}$$

The proposed method then coincides on meshes matching the interface with the harmonically-weighted discontinuous Galerkin method of [3]. Then, noting that $\alpha_S^{\mathrm{in/ex}}\leqslant 1$, we have

$$\frac{1}{2}\int_S \{\partial_{n,k}u_h\}^2_{\alpha_S} \leqslant \left(\alpha_S^{\mathrm{in}}\right)^2\int_S k_{\mathrm{in}}^2|\partial_n u_h|^2_{K^{\mathrm{in}}} + \left(\alpha_S^{\mathrm{ex}}\right)^2\int_S k_{\mathrm{ex}}^2|\partial_n u_h|^2_{K^{\mathrm{ex}}}$$
$$\leqslant \left(\alpha_S^{\mathrm{in}}\right)^2\frac{k_{\mathrm{in}}|S|}{|K^{\mathrm{in}}|}\int_{K^{\mathrm{in}}} k_{\mathrm{in}}|\nabla u_h|^2 + \left(\alpha_S^{\mathrm{ex}}\right)^2\frac{k_{\mathrm{ex}}|S|}{|K^{\mathrm{ex}}|}\int_{K^{\mathrm{ex}}} k_{\mathrm{ex}}|\nabla u_h|^2$$
$$\leqslant \frac{k_{\mathrm{in}}k_{\mathrm{ex}}|S|}{k_{\mathrm{ex}}|K^{\mathrm{in}}|+k_{\mathrm{in}}|K^{\mathrm{ex}}|}\int_{K^{\mathrm{in}}} k_{\mathrm{in}}|\nabla u_h|^2 + \frac{k_{\mathrm{in}}k_{\mathrm{ex}}|S|}{k_{\mathrm{ex}}|K^{\mathrm{in}}|+k_{\mathrm{in}}|K^{\mathrm{ex}}|}\int_{K^{\mathrm{ex}}} k_{\mathrm{ex}}|\nabla u_h|^2 \leqslant \frac{\gamma_S}{\hat{\gamma}}\int_K k|\nabla u_h|^2.$$

This estimate leads to uniform stability with respect to the diffusion parameters and cut-cell-geometry. Following the lines of analysis of [4], we can derive optimal-order error estimates of the modified NXFEM.

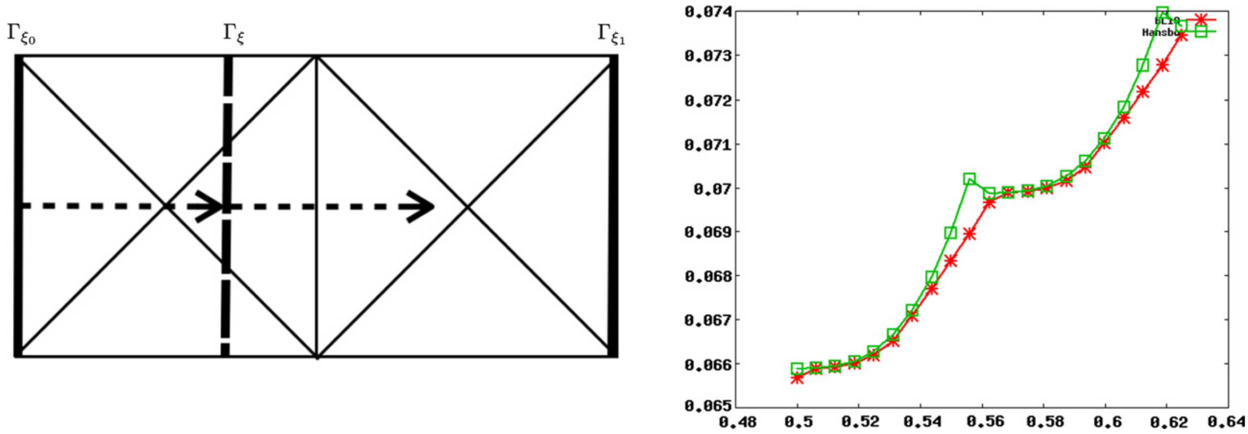

Fig. 2. Interface position (zoom in the mesh, left) and comparison of the energy norm error of the two methods (right).

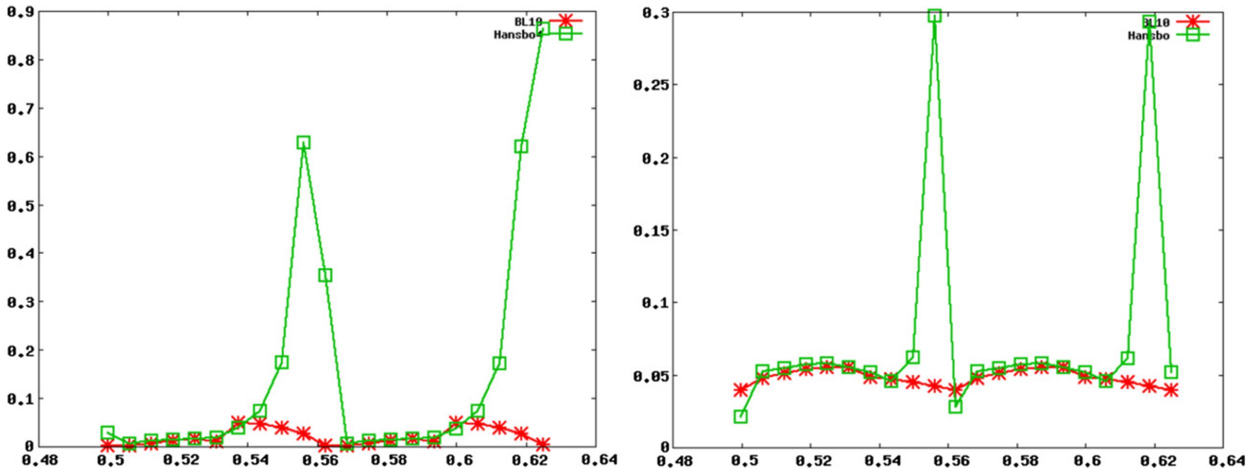

Fig. 3. Comparison of the two methods for varying position of the interface: error in flux (left) and jump in flux (right).

4. Numerical test

We consider a problem on $\Omega =]0,1[\times]0,1[$ with a straight interface $\Gamma_\xi := \{\xi\} \times [0,1]$. In order to study the robustness of the method we move the interface from $\xi_0 = 0.49999$ to $\xi_1 = 0.6249$ and choose $k_1 = 0.1$ and $k_2 = 10\,000$, see Fig. 2 for a zoom, on a fixed mesh not aligned with the interface. The data are chosen such that the function defined by $u = x^2/k_1$ for $x \leqslant \xi$ and $u = (x^2 - \xi^2)/k_2 + \xi^2/k_1$ else is an exact solution. This requires some trivial modifications to take into account the non-homogenous Dirichlet boundary condition.

In Fig. 2 we compare the error in energy norm and the jump of the discrete solution for the two methods. Since the exact solution depends on the interface position, both methods show a slightly increasing error when the interface moves to the right. Both methods show a robust behavior and the differences in energy error are relatively small, but increase when the interface approaches a mesh line.

Fig. 3 shows the $L^2(\Gamma)$-error in the flux, $\|\partial_{n_k} u - \partial^*_{n,k} u_h\|_S$ and the jump of the discrete flux $\|[\partial_{n,k} u_h]\|_S$. The difference between the two methods becomes quite apparent. If the interface approaches a mesh line, the original NXFEM leads to large errors, in contrast to the modifies one.

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