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Algebra

Equivalent condition for approximately Cohen–Macaulay complexes

Condition équivalente pour les complexes approximativement Cohen–Macaulay

Michał Lasoń [a,b]

[a] *Institute of Mathematics of the Polish Academy of Sciences, Śniadeckich 8, 00-956 Warszawa, Poland*
[b] *Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, 30-348 Kraków, Poland*

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ABSTRACT

We give a necessary and sufficient condition for a simplicial complex to be approximately Cohen–Macaulay. Namely it is approximately Cohen–Macaulay if and only if the ideal associated to its Alexander dual is componentwise linear and generated in two consecutive degrees. This completes the result of J. Herzog and T. Hibi who proved that a simplicial complex is sequentially Cohen–Macaulay if and only if the ideal associated to its Alexander dual is componentwise linear.

RÉSUMÉ

Nous donnons une condition nécessaire et suffisante pour qu'un complexe simplicial soit approximativement Cohen–Macaulay. Précisément, un complexe est approximativement Cohen–Macaulay si et seulement si l'idéal associé à son dual d'Alexander est engendré en deux degrés consécutifs et chacune de ses composantes a une résolution linéaire. Cela complète le résultat de J. Herzog et T. Hibi, qui démontrent qu'un complexe simplicial est séquentiellement Cohen–Macaulay si et seulement si chacune des composantes de l'idéal associé à son dual d'Alexander a une résolution linéaire.

1. Introduction

In [3] J. Eagon and V. Reiner proved that a simplicial complex is Cohen–Macaulay if and only if the ideal associated to its Alexander dual has linear resolution. Later J. Herzog and T. Hibi [5] generalized it and proved that a simplicial complex is sequentially Cohen–Macaulay if and only if the ideal associated to its Alexander dual is componentwise linear. We use their result to give a similar equivalent condition for a simplicial complex to be approximately Cohen–Macaulay.

We begin with a brief introduction to the topic. When we say that a simplicial complex is Cohen–Macaulay, sequentially Cohen–Macaulay, or approximately Cohen–Macaulay, we always think that its Stanley–Reisner ring has this property.

Definition 1. For a simplicial complex Δ on the set of vertices $\{1, \dots, n\}$ and a field $\mathbb{K}$, the *Stanley–Reisner ring* (or *face ring*) is the ring $\mathbb{K}[x_1, \dots, x_n]/I_\Delta = \mathbb{K}[\Delta]$, where I_Δ is generated by all squarefree monomials $x_{i_1} \cdots x_{i_l}$ for which $\{i_1, \dots, i_l\}$ is not a face in Δ.

E-mail address: michalason@gmail.com.

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We recall combinatorial description of Cohen–Macaulay complexes:

Definition 2. Let σ be a simplex in a simplicial complex Δ. The *link* of σ in Δ, denoted by $lk_{\Delta}\sigma$, is the simplicial complex $\{\tau \in \Delta\colon\ \tau \cap \sigma = \emptyset \text{ and } \tau \cup \sigma \in \Delta\}$.

Theorem 1. *(See Reisner [7].) Let $R = \mathbb{K}[\Delta]$ be the face ring of Δ. Then the following conditions are equivalent*:

(1) *R is Cohen–Macaulay ring.*
(2) *$\tilde{H}_i(lk_{\Delta}\sigma) = 0$ if $i < \dim(lk_{\Delta}\sigma)$ for all simplices $\sigma \in \Delta$.*

For some techniques of counting homology we refer the reader to Section 3.2 of [6]. We need also the following definitions:

Definition 3. (See [4].) A non-Cohen–Macaulay local ring A is called *approximately Cohen–Macaulay* if there is an element a in the maximal ideal such that $A/(a^n)$ is Cohen–Macaulay ring of dimension $\dim(A) - 1$ for all $n > 0$.

Definition 4. A ring A of dimension d is called *sequentially Cohen–Macaulay* if there exists a filtration of ideals of A:

$$0 = D_0 \subset D_1 \subset \cdots \subset D_t = A$$

such that each D_i/D_{i-1} is Cohen–Macaulay and

$$0 < \dim(D_1/D_0) < \dim(D_2/D_1) < \cdots < \dim(D_t/D_{t-1}) = d.$$

Definition 5. Let Δ be a simplicial complex on the set of vertices V, we define its *Alexander dual* to be $\Delta^* = \{V \setminus \sigma\colon\ \sigma \notin \Delta\}$.

Definition 6. We say that a graded ideal $I \subset A$ is *componentwise linear* if I_j has linear resolutions for each degree j.

There is a nice description of approximately Cohen–Macaulay rings:

Proposition 1. *(See [2].) Let A be a non-Cohen–Macaulay local ring of dimension d. Then the following conditions are equivalent*:

(1) *A is an approximately Cohen–Macaulay ring.*
(2) *A is a sequentially Cohen–Macaulay ring with filtration $0 = D_0 \subset D_1 \subset D_2 = A$, where $\dim(D_1) = d - 1$.*

2. Equivalent condition

We will make use of the following result of J. Herzog and T. Hibi [5]:

Theorem 2. *(See [5].) Let Δ be a simplicial complex. Then Stanley–Reisner ring $\mathbb{K}[\Delta]$ is sequentially Cohen–Macaulay if and only if I_{Δ^*}, the ideal associated to its Alexander dual, is componentwise linear.*

Our theorem reads as follows:

Theorem 3. *Let Δ be a simplicial complex. Then the Stanley–Reisner ring $\mathbb{K}[\Delta]$ is approximately Cohen–Macaulay if and only if I_{Δ^*}, ideal associated to its Alexander dual, is componentwise linear and generated in two consecutive degrees.*

Proof. By Proposition 1, $\mathbb{K}[\Delta]$ is approximately Cohen–Macaulay if and only if $\mathbb{K}[\Delta]$ is a sequentially Cohen–Macaulay ring with filtration

$$0 = D_0 \subset D_1 \subset D_2 = \mathbb{K}[\Delta],$$

where $\dim(D_1) = d - 1$. Due to the Theorem 2 of Herzog and Hibi this is equivalent to componentwise linearity of I_{Δ^*}, and existence of a filtration

$$0 = D_0 \subset D_1 \subset D_2 = \mathbb{K}[\Delta],$$

where $\dim(D_1) = d - 1$. From appendix of [1] we get that if such a filtration exists, then it is unique and coincides with the one given by

$$0 = M_0 \subset \cdots \subset M_{i-1} = I_{\Delta,\Delta^{\langle j_i - 1\rangle}} \subset \cdots \subset \mathbb{K}[\Delta],$$

where $I_{\Delta,\Delta^{\langle j_i-1\rangle}}$ is the ideal in $\mathbb{K}[\Delta]$ generated by monomials x_A, with $A \in \Delta \setminus \Delta^{\langle j_i-1\rangle}$. The simplicial complex $\Delta^{\langle j_i-1\rangle}$ is generated by faces of Δ of dimension at least j_i-1, where $j_1-1<\cdots<j_s-1$ are the dimensions of facets of Δ. We have also that $\dim(\Delta^{\langle j_i-1\rangle})=j_i-1$. Hence the desired filtration exists if and only if Δ has facets of dimension d and $d-1$. Ideal I_{Δ^*} is generated by monomials x_A for $A \notin \Delta^*$, that is, for $A=V\setminus\sigma$, where $\sigma\in\Delta$. We have to take all x_A corresponding to facets and they all already generate ideal. Hence the ideal I_{Δ^*} is generated in two consecutive degrees $v-d$ and $v-(d-1)$, where $|V|=v$. Since each step of our reasoning was an equivalence, the contrary also holds. □

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