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Partial quotients and representation of rational numbers ☆

Quotients partiels et représentation des nombres rationnels

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ABSTRACT

It is shown that there is an absolute constant C such that any rational $\frac{b}{q} \in]0,1[$, $(b,q)=1$, admits a representation as a finite sum $\frac{b}{q} = \sum_\alpha \frac{b_\alpha}{q_\alpha}$ where $\sum_\alpha \sum_i a_i(\frac{b_\alpha}{q_\alpha}) < C \log q$ and $\{a_i(x)\}$ denotes the sequence of partial quotients of x.

RÉSUMÉ

On démontre l'existence d'une constante C telle que tout rationnel $\frac{b}{q} \in]0,1[$, $(b,q)=1$, a une représentation comme somme finie $\frac{b}{q} = \sum_\alpha \frac{b_\alpha}{q_\alpha}$ où $\sum_\alpha \sum_i a_i(\frac{b_\alpha}{q_\alpha}) < C \log q$ et $\{a_i(x)\}$ est la suite des quotients partiels de x.

Version française abrégée

Cette Note résulte de résultats récents sur la conjecture de Zaremba obtenus dans [1] et quelques questions posées par R. Kenyon [4] sur la représentation de nombres rationnels comme somme de nombres rationnels dont les quotients partiels sont bornés par une constante absolue. Diverses propriétés de représentation de nombres réels x comme somme $x = y + z + \cdots$, où $y, z, \ldots$ ont leurs quotients partiels sujets à certaines bomes, ont en effet été établies (voir en particulier les résultats de M. Hall, [3]). Le problème de trouver des énoncés analogues pour les rationnels semble donc naturel. Dans cet esprit, on exploite ici les méthodes de [1] afin d'établir la propriété suivante :

Proposition 1. *Il existe une constante C telle que tout rationnel $\frac{b}{q} \in]0,1[$, $(b,q)=1$, admette une représentation comme somme finie $\frac{b}{q} = \sum_\alpha \frac{b_\alpha}{q_\alpha}$ où $\sum_\alpha \sum_i a_i(\frac{b_\alpha}{q_\alpha}) < C \log q$.*

1. Some background

It was shown by M. Hall [3] that every number in the interval $]\sqrt{2}-1, 4\sqrt{2}-4[$ is the sum of two continued fractions whose partial quotients do not exceed four (see [3], Theorem 3.1).

Recently, R. Kenyon brought to the author's attention the problem of obtaining a result in the flavor of Hall's theorem for the rational numbers. There are several possible formulations. One could ask for instance if there is an absolute constant C

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such that given $\frac{b}{q} \in \mathbb{Q}_+ \cap I$, I a suitable interval, there is a representation of b as a sum of at most C positive integers b_i such that each of the fractions $\frac{b_i}{q}$ has its partial quotients bounded by C. To be mentioned here is the (still unsolved) conjecture of Zaremba, according to which for all $q \in \mathbb{Z}_+$, there is some $(b, q) = 1$ such that $\frac{b}{q}$ has partial quotients bounded by five (or some absolute constant). Alternatively, one can ask if any element in $\mathbb{Q}_+ \cap I$ is sum of two (or at most C) rationals with partial quotients bounded by C. While we will leave these questions unanswered here, our aim is to prove the following property in a similar spirit:

Proposition 1. *There is an absolute constant C such that any rational $\frac{b}{q} \in]0, 1[$, $(b, q) = 1$, admits a representation as a finite sum*

$$\frac{b}{q} = \sum_\alpha \pm \frac{b_\alpha}{q_\alpha} \quad (b_\alpha, q_\alpha) = 1 \tag{1}$$

such that

$$\sum_\alpha \sum_i a_i\left(\frac{b_\alpha}{q_\alpha}\right) \leqslant C \log q \tag{2}$$

where $\{a_i(x)\}$ denotes the sequence of partial quotients of $x \in]0, 1[$.

As R. Kenyon points out, a statement of this kind may be viewed as a measure of complexity of rationals of given height. Note that since $\sum_i a_i(\frac{b_\alpha}{q_\alpha}) \gtrsim \log q_\alpha$, above estimate is essentially optimal.

2. Preliminaries

Our main analytical tools are the results and methods of the recent paper [1] on Zaremba's conjecture. It is shown in [1] that for a large enough constant A (we may take $A = 50$), for all $q \in \mathbb{Z}_+$ outside an exceptional set $E \subset \mathbb{Z}_+$ of zero-density, there is some $b \in \mathbb{Z}_+$, $(b, q) = 1$ such that

$$\frac{b}{q} \in \mathcal{R}_A = \left\{x \in \mathbb{Q} \cap [0, 1]\colon \max_i a_i(x) \leqslant A\right\}. \tag{3}$$

More quantitatively, one gets an estimate

$$\left|E \cap [1, N]\right| < N^{1-\frac{c}{\log\log N}}. \tag{4}$$

Instead of (4), it is possible to obtain a power saving

Proposition 2. *The above statement holds with E satisfying*

$$\left|E \cap [1, N]\right| < N^{1-c_1} \tag{5}$$

with $c_1 > 0$ some constant.

Recalling the approach from [1], elements $\frac{b}{q} \in \mathcal{R}_A$ are produced from elements $g = \begin{pmatrix} * & b \\ * & q \end{pmatrix}$ in the semi-group $\mathcal{G}_A$ generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} \quad (1 \leqslant a \leqslant A). \tag{6}$$

We use the Hardy–Littlewood circle method in order to analyze exponential sums of the form

$$\sum \lambda(g) e(g_{22}\theta) \tag{7}$$

with λ a suitable distribution on $\mathcal{G}_A$. Fixing some large N, the distribution λ is obtained from a product of certain Archimedian balls in $\mathcal{G}_A$. As usual, the circle method involves a treatment of minor and major arcs contributions and those depend on different ingredients. The estimates on minor arcs result from Vinogradov-type bilinear estimates, exploiting the multi-linear structure of λ. A precise evaluation of (7) on the major arcs (up to an error term) is possible using spectral methods. We use the thermodynamical approach and the results from [2] based on the theory of expansion in $SL_2(q)$. The error term in the counting and the size of the exceptional set in (4) depend on the width of the resonance free regions for the congruence transfer operators. It turns out that the gain of the $N^{-\frac{c}{\log\log N}}$-factor rather than N^{-c} comes from introducing balls $B_M = \{g \in \mathcal{G}_A\colon \|g\| \leqslant M\}$. If instead of balls we consider slightly more general distributions (obtained as average of balls over suitable radii), one may recover a full powergain N^{-c} (the cutoff-level for the major arcs may then be set at N^c for some $c > 0$, rather than $N^{\frac{c}{\log\log N}}$).

In order to prove Proposition 1, we need one more further refinement.

Proposition 3. *Taking again A sufficiently large, there is a constant $c > 0$ such that the following holds:*
Let $N \in \mathbb{Z}_+$ be large enough, $d \in \mathbb{Z}_+$, $d < N^c$ and $\beta \in \mathbb{Z}$, $(\beta, d) = 1$. There is a subset $E_{N;d,\beta} \subset \mathbb{Z} \cap [1, N]$ such that

$$|E_{N,d,\beta}| < N^{1-c} \tag{8}$$

and for all $q \in \mathbb{Z}_+ \setminus E_{N;d,\beta}$, $q < N$, there is $b \in \mathbb{Z}_+$, $b < q$, $(b, q) = 1$ satisfying

$$\frac{b}{q} \in \mathcal{R}_A \tag{9}$$

and

$$b \equiv \beta \pmod d. \tag{10}$$

Returning to [1], the incorporation of the additional congruence condition (10) is harmless at the level of the minor arcs estimates, provided d is sufficiently small. Of course the condition (10) enters the singular series in the treatment of the major arcs and the assumption $(\beta, d) = 1$ ensures that there are no local obstructions.

Obviously Proposition 3 implies that for some constant $c > 0$, the following holds:

Proposition 3′. *There is a subset $E_N \subset \mathbb{Z} \cap [1, N]$ such that*

$$|E_N| < N^{1-c} \tag{11}$$

and for all $q \in \mathbb{Z}_+ \setminus E_N$, $q < N$ and all $d \in \mathbb{Z}_+$, $d < N^c$, and $\beta \in \mathbb{Z}$, $(\beta, d) = 1$, there is some $b \in \mathbb{Z}_+$, $(b, q) = 1$ satisfying (9) and (10).

3. Proof of Proposition 1

Denote $C(\frac{b}{q})$ the minimum of the left hand side of (2) over all representations (1). First, observe that it suffices to show that

$$\frac{b}{q} = \frac{b'}{q'} + \frac{b''}{q''} \tag{12}$$

with

$$C\left(\frac{b'}{q'}\right) < C \log q \tag{13}$$

and

$$q'' < \sqrt{q} \tag{14}$$

with C in (13) some absolute constant. We may then indeed iterate.

Let $c > 0$ be the constant from Proposition 3′. Set

$$\delta = \frac{1}{10} c \quad \text{and} \quad r = \left[\delta^{-2}\right] + 1. \tag{15}$$

We claim that there are primes $p_1, \ldots, p_r$, $(p_i, q) = 1$, $p_i < q^\delta$ and $p_i \sim q^\delta$ satisfying the following two conditions:

$$qp_1 \ldots p_r \notin E_{q^{1+r\delta}} \quad \text{with } E_N \text{ as in Proposition 3}'. \tag{16}$$

$$\text{For all } I \subset \{1, \ldots, r\},\ I \neq \phi, \quad \prod_{i \in I} p_i \notin E_{q^{\delta |I|}}. \tag{17}$$

Indeed, consider all integers of the form $qp_1 \ldots p_r$ with p_i as above.

Their number is at least

$$\frac{q^{r\delta}}{(\log q)^r}. \tag{18}$$

On the other hand, by (11)

$$|E_{q^{1+r\delta}}| < q^{(1+r\delta)(1-c)} \leqslant q^{1+r\delta - c\delta^{-1}} < q^{r\delta - 1} = o\left(\frac{q^{r\delta}}{(\log q)^r}\right).$$

Next, consider condition (17) and fix $I \subset \{1, \dots, r\}$, $I \neq \phi$. Among the integers considered above, those for which $\prod_{i \in I} p_i \in E_{q^{|I|\delta}}$ account for at most

$$q^{(r-|I|)\delta} |E_{q^{\delta|I|}}| < q^{r\delta - c\delta} = o\left(\frac{q^{r\delta}}{(\log q)^r}\right).$$

Hence we may find $p_1, \dots, p_r$ with the desired properties.

Returning to (12)–(14), write with $p_1, \dots, p_r$ as above

$$\frac{b}{q} = \frac{bp_1 \dots p_r}{qp_1 \dots p_r}.$$

Since $(b, q) = (p_i, q) = 1$, $(bp_1 \dots p_r, q) = 1$ with $q < N^c$, $N = q^{1+r\delta}$.

Since (16) holds, Proposition 3′ implies that there is some $b_0 \in \mathbb{Z}_+$, $b_0 < qp_1 \dots p_r$ such that $\frac{b_0}{p_1 \dots p_r q} \in \mathcal{R}_A$ and $b_0 \equiv bp_1 \dots p_r \pmod q$. Hence

$$C\left(\frac{b_0}{p_1 \dots p_r q}\right) < C(A) \log(p_1 \dots p_r q) \leqslant C(A)(1 + r\delta) \log q = C \log q \tag{19}$$

and we may write

$$\frac{b}{q} = \frac{b_0}{qp_1 \dots p_r} + \frac{a_1}{\prod_{i \in I_1} p_i} \quad \text{with } a_1 \in \mathbb{Z} \text{ and } \left(a_1, \prod_{i \in I_1} p_i\right) = 1 \text{ if } I_1 \neq \phi. \tag{20}$$

If $\prod_{i \in I_1} p_i < \sqrt{q}$, set $\frac{b''}{q''} = \frac{a_1}{\prod_{i \in I_1} p_i}$ in (12).

If $\prod_{i \in I_1} p_i \geqslant \sqrt{q}$, use (17) and take $d = p_{i_1}$, i_1 chosen from I_1, $\beta = a_1$.

By definition of δ, $p_{i_1} < q^\delta < (q^{\frac{1}{2}})^c$ and we get some $b_1 \equiv a_1 \pmod{p_{i_1}}$ with $\frac{b_1}{\prod_{i \in I_1} p_i} \in \mathcal{R}_A$. Thus

$$C\left(\frac{b_1}{\prod_{i \in I_1} p_i}\right) < C(A) \log\left(\prod_{i \in I_1} p_i\right) < C \log q \tag{21}$$

and

$$\frac{a_1}{\prod_{i \in I_1} p_i} = \frac{b_1}{\prod_{i \in I_1} p_i} + \frac{a_2}{\prod_{i \in I_2} p_i} \quad \text{with } I_2 \subset I_1 \setminus \{i_1\},\ \left(a_2, \prod_{i \in I_2} p_i\right) = 1 \text{ if } I_2 \neq \phi. \tag{22}$$

The continuation of the process is clear and it terminates after at most r steps, leading to a representation

$$\frac{b}{q} = \frac{b_0}{qp_1 \dots p_r} + \frac{b_1}{\prod_{i \in I_1} p_i} + \dots + \frac{b_\rho}{\prod_{i \in I_\rho} p_i} + \frac{b''}{q''} = \frac{b'}{q'} + \frac{b''}{q''} \tag{23}$$

satisfying (13), (14).

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References

[1] J. Bourgain, K. Kontorovich, On Zaremba's conjecture, preprint, 2011, arXiv:1107.3776v1.
[2] J. Bourgain, A. Gamburd, P. Sarnak, Generalization of Selberg's 3/16 theorem and affine sieve, Acta Math. 207 (2) (2011) 255–290.
[3] M. Hall, On the sum and product of continued fractions, Annals of Math. 48 (4) (1947).
[4] R. Kenyon, private communication.