Partial Differential Equations/Optimal Control

A constructive method for the stabilization of the wave equation with localized Kelvin–Voigt damping

Une méthode constructive pour la stabilisation de l’équation des ondes avec un amortissement localisé de type Kelvin–Voigt

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A R T I C L E  I N F O

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A B S T R A C T

We consider the wave equation with Kelvin–Voigt damping in a bounded domain. The damping is localized in a suitable open subset of the domain under consideration. The exponential stability result proposed by Liu and Rao for that system assumes that the damping is localized in a neighborhood of the whole boundary, and the damping coefficient is continuously differentiable with a bounded Laplacian. We propose a new solution to the exponential stability problem based on the introduction of a new variable, and a constructive frequency domain approach. The main features of our method are: (i) the damping region need not be a neighborhood of the whole boundary; (ii) the damping coefficient is assumed to be bounded measurable with bounded measurable gradient only; (iii) the introduction of a new variable. These features enable us to improve on the damping coefficient smoothness and more especially on the feedback control region. Further, when combined with a recent result of Borichev and Tomilov on the polynomial decay of bounded semigroups, the new method enables us to prove a polynomial decay estimate of the energy when the damping coefficient is bounded measurable only.

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R É S U M É

On considère l’équation des ondes avec un amortissement de type Kelvin–Voigt dans un domaine borné. L’amortissement est localisé dans un sous-ensemble ouvert convenablement choisi dans le domaine en question. Le résultat de stabilité exponentielle proposé par Liu et Rao pour ce système suppose que l’amortissement est localisé dans un voisinage de tout le bord, et le coefficient de l’amortissement est continûment dérivable avec un laplacien borné. Nous proposons une solution nouvelle au problème de la stabilité exponentielle basée sur l’introduction d’une nouvelle variable et une méthode constructive de type domaine des fréquences. Les caractéristiques principales de notre approche sont : (i) il n’est pas nécessaire que la région où l’amortissement est localisé soit un voisinage de tout le bord ; (ii) le coefficient d’amortissement ainsi que son gradient sont supposés seulement bornés et mesurables ; (iii) l’introduction d’une nouvelle variable. Ces éléments nous permettent d’améliorer la régularité du coefficient d’amortissement, et plus particulièrement la région du contrôle dissipatif. De plus, si on combine la nouvelle méthode avec un résultat récent de Borichev et Tomilov sur la décroissance polynômiale de semi-groupes,

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The energy $E$ associated to (1) is given by

$$E(t) = \frac{1}{2} \int_{\Omega} \left( |y_t(x,t)|^2 + |\nabla y(x,t)|^2 \right) \, dx.$$  

(3)

The energy $E$ is a nonincreasing function of the time variable $t$ as we have for almost every $t \geq 0$,

$$E'(t) = -\int_{\Omega} a(x) |\nabla y(x,t)|^2 \, dx.$$  

(4)
Setting \( Z = (\varphi, y) \), (2) may be recast as: \( Z' - \mathcal{A}Z = 0 \) in \( (0, \infty) \), \( Z(0) = (\varphi^0, y^0) \), where the unbounded operator \( \mathcal{A} \) is given by

\[
\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \Delta & \text{div}(\mathbf{a} \nabla \cdot) \end{pmatrix}
\]

with \( D(\mathcal{A}) = \{(u, v) \in H^1_0(\Omega) \times H^1_0(\Omega); \Delta u + \text{div}(\mathbf{a} \nabla v) \in L^2(\Omega)\} \).

Introduce the Hilbert space over the field \( \mathbb{C} \) of complex numbers \( \mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \), equipped with the norm

\[
\| Z \|^2_{\mathcal{H}} = \int_\Omega \left( |v|^2 + |\nabla u|^2 \right) \, dx, \quad \forall Z = (u, v) \in \mathcal{H}.
\]

We now introduce a geometric constraint (GC) on the subset \( \omega \) where the dissipation is effective; we proceed as in Liu [12] (see also [8,11]).

(GC). There exist open sets \( \Omega_j \subset \Omega \) with piecewise smooth boundary \( \partial \Omega_j \), and points \( x_j^0 \in \mathbb{R}^N \), \( j = 1, 2, \ldots, J \), such that \( \Omega_i \cap \Omega_j = \emptyset \), for any \( 1 \leq i < j \leq J \), and:

\[
\Omega \cap \mathcal{N}_\delta \left[ \left( \bigcup_{j=1}^J \Gamma_j \right) \cup \left( \Omega \setminus \bigcup_{j=1}^J \Omega_j \right) \right] \subset \omega,
\]

for some \( \delta > 0 \), where \( \mathcal{N}_\delta(S) = \bigcup_{y \in S} \{ y \in \mathbb{R}^N; |y - \bar{y}| < \delta \} \), for \( S \subset \mathbb{R}^N \), \( \Gamma_j = \{ x \in \partial \Omega_j; (x - x_j^0) \cdot v_j(x) > 0 \} \), \( v_j \) being the unit normal vector pointing into the exterior of \( \Omega_j \).

Our main results read:

**Theorem 1.1.** Suppose that \( \omega \) is an arbitrary nonempty open set in \( \Omega \). Let the damping coefficient \( a \) be bounded measurable, and positive in \( \omega \). The operator \( \mathcal{A} \) generates a \( C_0 \) semigroup of contractions \( (S(t))_{t \geq 0} \) on \( \mathcal{H} \), which is strongly stable:

\[
\lim_{t \to \infty} \| S(t) Z^0 \|_{\mathcal{H}} = 0, \quad \forall Z^0 \in \mathcal{H}.
\]

Furthermore, if \( a \) satisfies (1), and \( \omega \) satisfies the geometric constraint (GC) above, then we have the polynomial decay estimate:

\[
\exists C_0 > 0: \| S(t) Z^0 \|_{\mathcal{H}} \leq \frac{C_0 \| Z^0 \|_{D(\mathcal{A})}}{(1 + t)^{\frac{1}{2}}}, \quad \forall t \geq 0, \forall Z^0 \in D(\mathcal{A}).
\]

**Theorem 1.2.** Suppose that \( \omega \) satisfies the geometric constraint (GC) above. As for the damping coefficient \( a \), further assume that \( a \in \mathcal{W}^{1,\infty}(\Omega) \) with \( |\nabla a(x)|^2 \leq M_0 a(x) \), for almost every \( x \in \Omega \), for some positive constant \( M_0 \). The semigroup \( (S(t))_{t \geq 0} \) is exponentially stable, viz., there exist positive constants \( M \) and \( \lambda \) with

\[
\| S(t) Z^0 \|_{\mathcal{H}} \leq M \exp(-\lambda t) \| Z^0 \|_{\mathcal{H}}, \quad \forall t \geq 0, \forall Z^0 \in \mathcal{H}.
\]

**Remark 1.3.** The second part of Theorem 1.1 on the polynomial decay of the energy is in sharp contrast with what happens in the case of a viscous damping of the form \( a_\nu \partial_t \); in fact, when (GC) holds, the geometric control condition of Bardos, Lebeau and Rauch [1] is met, and exponential decay of the energy should be expected. However, it was shown in the one-dimensional setting that exponential decay of the energy fails if the coefficient \( a \) is discontinuous along the interface [13]; one would expect the same conclusion to hold in the multidimensional setting, but this is not done yet.

The earlier result in the literature establishing the exponential decay of the energy in the multidimensional setting assumes that \( \omega \) is a neighborhood of the whole boundary, and the damping coefficient \( a \) satisfies [14,15]: \( a \in C^{1,1}(\bar{\Omega}) \), \( \Delta a \in L^\infty(\Omega) \), and \( |\nabla a(x)|^2 \leq M_0 a(x) \), for almost every \( x \in \Omega \), for some positive constant \( M_0 \). It can be checked that if \( (u, v) \) lies in \( D(\mathcal{A}) \), then \( u \) lies in \( H^2_0(\Omega) \) only, as opposed to \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) in the case of frictional damping \( a_\nu \); this lost of derivative makes it very difficult to apply the usual multiplier technique. Before doing so, a change of variable is needed; the change of variable in [15] led its authors to impose more constraints on the damping coefficient and especially on the feedback control region than necessary. In this regard, Theorem 1.2 strongly improves the exponential stability result of [14,15].

2. Ideas for proving Theorems 1.1 and 1.2

The proof of the first part of Theorem 1.1 about the semigroup generation, and its strong stability may be found in [15]. Additionally it is also shown in [15] that the imaginary axis is contained in the resolvent set \( \rho(\mathcal{A}) \) of \( \mathcal{A} \). Thanks to a recent result [2, Theorem 2.4], the proof of the second part then amounts to showing the resolvent estimate.
\[ (ibI - A)^{-1} \|_{\mathcal{L}(H)} = O(|b|^2) \] as \(|b| \to +\infty\). To this end, let \( U \in H \), and let \( b \) be a real number with \(|b| \geq 1\). Since the range of \( ibI - A \) is \( H \), there exists \( Z \in D(A) \) such that
\[ ibZ - AZ = U. \] \hfill (10)
We will sketch the proof of the fact that
\[ \|Z\|_H \leq K_0|b|^2\|U\|_H, \] \hfill (11)
where here and in the sequel, \( K_0 \) is a generic positive constant that may eventually depend on \( \Omega, \omega, \) and \( a \), but not on \( b \).

To establish (11), first, we note that if \( Z = (u, v) \), and \( U = (f, g) \), then (10) may be recast as
\[ ibu - v = f, \]
\[ ibv - \Delta u - \text{div}(a\nabla v) = g. \] \hfill (12)
Taking the inner product with \( Z \) on both sides of (10), then taking the real parts, we immediately get
\[ |\sqrt{a}\nabla v|^2 \leq \|U\|_H\|Z\|_H. \] \hfill (13)
It now follows from the first equation in (12), and (13):
\[ b^2|\sqrt{a}\nabla u|^2 \leq 2|\sqrt{a}\nabla f|^2 + 2|\sqrt{a}\nabla v|^2 + K_0\|f\|_{H^1_0(\Omega)}^2 \leq 2\|U\|_H\|Z\|_H + K_0\|U\|_H^2. \] \hfill (14)
Introduce \( u_1 = u - w \), where \( w = G(\text{div}(a\nabla v)) \), with \( G \in \mathcal{L}(H^{-1}(\Omega); H^1_0(\Omega)) \) being the inverse of the negative Laplacian with Dirichlet boundary conditions. Since \((u, v)\) lies in \( D(A)\), one easily checks that \( u_1 \in H^2(\Omega) \cap H^1_0(\Omega) \). Thanks to (13), we note that
\[ \|w\|_{H^1_0(\Omega)} \leq |a|\|U\|_H\|Z\|_H, \]
\[ \|u_1\|_{H^1_0(\Omega)} \leq \|Z\|_H + \sqrt{|a|\|U\|_H\|Z\|_H}. \] \hfill (15)
On the other hand, the second equation in (12) becomes
\[ ibv - \Delta u_1 = g. \] \hfill (16)
It immediately follows from (16):
\[ |b|\|v\|_{H^{-1}(\Omega)} \leq \|u_1\|_{H^1_0(\Omega)} + K_0|g|_2. \] \hfill (17)
Now for each \( j = 1, \ldots, J \), where \( J \) appears in the geometric constraint (GC) stated above, set \( m^j(x) = x - x_j^0 \) and \( R_j = \sup(|m^j(x)|, x \in \Omega) \). Let \( 0 < \delta_0 < \delta_1 < \delta \), where \( \delta \) is the one given in (GC). Set:
\[ S = \left( \bigcup_{j=1}^J R_j \right) \cup \left( \Omega \setminus \bigcup_{j=1}^J R_j \right), \quad Q_0 = N_{\delta_0}(S), \quad Q_1 = N_{\delta_1}(S), \quad \omega = \Omega \cap Q_1, \]
and for each \( j \), let \( \psi_j \) be a function satisfying \( \psi_j \in W^{1,\infty}(\Omega), 0 \leq \psi_j \leq 1, \psi_j = 1 \) in \( \Omega \setminus Q_j, \psi_j = 0 \) in \( \Omega \cap Q_0 \).

Let \( \alpha > 0 \) and \( \beta \) be real constants with \( \alpha(N-2) < \beta < \alpha N \). Multiply (16) by \( \bar{u}_1 \), integrate on \( \Omega \), and take real parts to get
\[ \beta \Re \int_\Omega \bar{u}_1 dx = \beta \Re \int_\Omega (ibv - \Delta u_1)\bar{u}_1 dx = \beta \|u_1\|_{H^1_0(\Omega)}^2 + \beta \Re \int_\Omega v(ibu - ib\bar{w}) dx. \] \hfill (18)
Using (12), it follows: \( \beta \Re \int_\Omega v(ibu - ib\bar{w}) dx = \beta \Re \int_\Omega v(-\bar{v} - \bar{f} - ib\bar{w}) dx \). Hence
\[ \beta \Re \int_\Omega \bar{u}_1 dx = \beta \|u_1\|_{H^1_0(\Omega)}^2 - \beta |v|^2 - \beta \Re \int_\Omega v(\bar{f} + ib\bar{w}) dx. \] \hfill (19)
It follows from (15) and (17): \( |\beta \Re \int_\Omega v(\bar{f} + ib\bar{w}) dx| \leq K_0(\|U\|_H\|Z\|_H + \|U\|_{H^3_0(\Omega)}^2 \|Z\|_H^2 + \|U\|_{H^3_0(\Omega)}^2 \|Z\|_H^2). \) Whence
\[ K_0(\|U\|_H\|Z\|_H + \|U\|_{H^3_0(\Omega)}^2 \|Z\|_H^2 \|Z\|_H^2 + \|U\|_{H^3_0(\Omega)}^2 \|Z\|_H^2) \geq \beta \|u_1\|_{H^1_0(\Omega)}^2 - \beta |v|^2. \] \hfill (20)
Now, multiply (16) by \( 2\alpha \psi_j m^j \cdot \nabla \bar{u}_1 \), integrate on \( \Omega_j \), and take real parts to get
\[ 2\alpha \Re \int_{\Omega_j} g(\psi_j m^j \cdot \nabla \bar{u}_1) dx = 2\alpha \Re \int_{\Omega_j} \psi_j m^j \cdot \nabla (-\bar{v} - \bar{f} - ib\bar{w}) dx - 2\alpha \Re \int_{\Omega_j} \Delta u_1(\psi_j m^j \cdot \nabla \bar{u}_1) dx. \] \hfill (21)
The application of Green’s formula shows
\[
-2\alpha\Re\int_{\Omega_j} \nabla \varphi_j \cdot \nabla \tilde{v} \, dx = \alpha N \int_{\Omega_j} \varphi_j |v|^2 \, dx + \alpha \int_{\partial \Omega_j} (m^1 \cdot \nabla \varphi_j) |v|^2 \, d\Gamma' + 2\alpha \int_{\partial \Omega_j} \varphi_j (m^1 \cdot v^j) |v|^2 \, d\Gamma',
\]
(22)
and
\[
-2\alpha\Re\int_{\Omega_j} \Delta u_1 (\varphi_j m^1 \cdot \nabla \tilde{u}_1) \, dx = 2\alpha\Re\int_{\Omega_j} (\nabla u_1 \cdot \nabla \varphi_j) m^1 \cdot \nabla \tilde{u}_1 \, dx - (N-2)\alpha \int_{\Omega_j} \varphi_j |\nabla u_1|^2 \, dx
- \alpha \int_{\Omega_j} (m^1 \cdot \nabla \varphi_j) |\nabla u_1|^2 \, dx + \alpha \int_{\partial \Omega_j} \varphi_j (m^1 \cdot v^j) |\nabla u_1|^2 \, d\Gamma'
- 2\alpha\Re\int_{\partial \Omega_j} \left( \frac{\partial u_1}{\partial v^j} \right) \varphi_j m^1 \cdot \nabla \tilde{u}_1 \, d\Gamma'.
\]
(23)
It can be checked that the boundary integral in (22) vanishes while
\[
\alpha \int_{\partial \Omega_j} \varphi_j (m^1 \cdot v^j) |\nabla u_1|^2 \, d\Gamma' - 2\alpha\Re\int_{\partial \Omega_j} \left( \frac{\partial u_1}{\partial v^j} \right) \varphi_j m^1 \cdot \nabla \tilde{u}_1 \, d\Gamma' \geq 0.
\]
Hence, taking the sum over \(j\), one derives from (21)–(23), and (1):
\[
K_0 \left( \|U\|_{1,\Omega} \|Z\|_{\mathcal{H}} + \frac{1}{2} \|U\|_{1,\Omega}^2 \|Z\|_{\mathcal{H}}^2 \right) \geq \alpha N \int_{\Omega_j} |v|^2 \, dx - (N-2)\alpha \|u_1\|_{H^1_0(\Omega)}^2 - K_0 \int_{\Omega} a|\nabla v|^2 \, dx
- K_0 \int_{\Omega_j} a|\nabla u_1|^2 \, dx - 2\alpha\Re\int_{\Omega_j} \sum_{j=1}^J v \varphi_j m^1 \cdot \nabla \tilde{w} \, dx.
\]
(24)
The estimates established so far are conducted the same way in the proofs of Theorems 1.1 and 1.2. Only estimating the utmost right term in the lower line of (24) requires different approaches. In the case of Theorem 1.1, one easily derives, thanks to (15):
\[
2\alpha\Re\int_{\Omega_j} \sum_{j=1}^J v \varphi_j m^1 \cdot \nabla \tilde{w} \, dx \leq K_0 |b| \|U\|_{1,\Omega} \|Z\|_{\mathcal{H}} \|\tilde{Z}\|_{\mathcal{H}} \|\tilde{w}\|_{\mathcal{H}}.
\]
(25)
Combining (1), (14), (20), (24), (25), and using Young inequality, one derives (11). Applying [2, Theorem 2.4], one gets the claimed polynomial decay estimate, thereby completing the proof sketch of Theorem 1.1. As for Theorem 1.2, estimating that term is more intricate; this is where the smoothness assumption on the damping coefficient \(a\) is needed. Before proceeding with that estimate, we note that we shall now show that for \(U, Z\) as above, and all \(b\), one has
\[
\|Z\|_{\mathcal{H}} \leq K_0 \|U\|_{\mathcal{H}},
\]
(26)
from which will follow the claimed exponential decay estimate thanks to, e.g. [7, Theorem 3], or [17, Corollary 4]. To this end, and as noted above, it remains to estimate the utmost right term in the lower line of (24). For each \(j\), introduce \(z_j \in H^1_0(\Omega)\) with \(-\Delta z_j = \text{div}(\varphi_j m^1 v)\) in \(\Omega\). One checks that for each \(j\)
\[
\Re\int_{\Omega_j} \varphi_j \nu \cdot \nabla \bar{v} \, dx = -\Re\int_{\Omega_j} a \varphi_j \nu \cdot \nabla \bar{v} \, dx - \Re\int_{\Omega_j} (\nabla \nu \cdot \nabla z_j) \bar{v} \, dx.
\]
(27)
from which one derives
\[
2\alpha\Re\int_{\Omega_j} \sum_{j=1}^J v \varphi_j m^1 \cdot \nabla \tilde{w} \, dx \leq K_0 |b| \|\sqrt{a}v\|_{L^2(\Omega)} \left( \|U\|_{1,\Omega} \|Z\|_{\mathcal{H}} \|\tilde{Z}\|_{\mathcal{H}} \|\tilde{w}\|_{\mathcal{H}} \right).
\]
(28)
Now, multiply the second equation in (12) by \(-iba\nu\), apply Green’s theorem over \(\Omega\), use the first equation in (12), and take real parts to derive
\[
b^2 \|\sqrt{a}v\|_{L^2}^2 = \Re \int_{\Omega} (\nabla(v + f) \cdot \nabla(a\nu) - iba(v \cdot \nabla)\nu - ibga\bar{v}) \, dx.
\]
Applying Hölder inequality, then Young inequality, we finally get
\[
b^2 \|\sqrt{a}v\|_{L^2}^2 \leq K_0 \left( \|U\|_{\mathcal{H}} \|Z\|_{\mathcal{H}} + \frac{1}{2} \|U\|_{1,\Omega}^2 \|Z\|_{\mathcal{H}}^2 + \frac{1}{2} \|U\|_{1,\Omega} \|Z\|_{\mathcal{H}} + \|U\|_{1,\Omega} \|Z\|_{\mathcal{H}}^2 \right).
\]
(29)
Reporting (29) in (28), we find
\[
2\alpha\|\Re b\|\sum_{j=1}^{N} \int v\varphi_{j}m' \cdot \nabla \hat{w} \, dx \leq K_0 \left( \|U\|_{H^1} \|Z\|_{H^1} + \|U\|_{H^3}^{\frac{3}{2}} \|Z\|_{H^1} + \|U\|_{H^3}^{\frac{5}{4}} \|Z\|_{H^1}^{\frac{7}{4}} \right).
\]
(30)

The combination of (1), (14), (20), (24), (30), and the application of Young inequality lead to (26) for all \(b\) with \(|b| \geq 1\). Next, using the continuity of the resolvent operator yields (26) for all \(b\), thereby completing our proof sketch.

References