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Unramified cohomology, $\mathbb{A}^1$-connectedness, and the Chevalley–Warning problem in Grothendieck ring ☆

Cohomologie non ramifiée, $\mathbb{A}^1$-connexité et le problème de Chevalley–Warning dans l'anneau de Grothendieck

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ABSTRACT

We study the Chevalley–Warning problem in the Grothendieck ring $K_0(Var/k)$. We show that the $\mathbb{A}^1$-homotopy theory yields well-defined invariants on $K_0(Var/k)/\mathbf{L}$, in particular the Brauer group is such an invariant. We use this to give a concrete counter-example to the Chevalley–Warning conjecture over a C_1-field (Brown and Schnetz, 2011 [6]). This also gives a negative answer to the question in Bilgin (2011) [5, Ques. 3.8].

RÉSUMÉ

Nous étudions le problème de Chevalley–Warning dans l'anneau de Grothendieck $K_0(Var/k)$. Nous montrons que la théorie $\mathbb{A}^1$-homotopie fournit des invariants sur $K_0(Var/k)/\mathbf{L}$. En particulier le groupe de Brauer est un tel invariant. Nous utilisons cela pour donner un contre-exemple concret à la conjecture de Chevalley–Warning sur un corps C_1 (Brown et Schnetz, 2011 [6]). Cela donne aussi une réponse négative à la question dans Belgin (2011) [5, Ques. 3.8].

1. Introduction

Let k be a field and Var/k be the category of varieties over k. We denote by $K_0(Var/k)$ the Grothendieck ring of varieties over k. Over a finite field $k = \mathbb{F}_q$, the Chevalley–Warning theorem (cf. [3]) states that a projective hypersurface $X \subset \mathbb{P}^n$ of degree $d \leqslant n$ satisfies the congruence formula

$$\left|X(\mathbb{F}_q)\right| \equiv 1 \mod q. \tag{1}$$

The counting point $X \mapsto |X(\mathbb{F}_q)|$ gives rise to a ring homomorphism

$$|-|: K_0(Var/\mathbb{F}_q) \to \mathbb{Z},$$

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from which one may reformulate the congruence formula (1) as $|[X]| \equiv 1 \bmod |\mathbf{L}|$, where we denote by $\mathbf{L}$ the class of the affine line $[\mathbb{A}^1]$ in $K_0(Var/\mathbb{F}_q)$. The geometric Chevalley–Warning problem for smooth projective hypersurfaces concerns with the following question:

Question 1.1. Let k be a field and $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $\leqslant n$ such that $X(k) \neq \emptyset$. Whether is it true that $[X] \equiv \mathbf{1} \bmod \mathbf{L}$ in $K_0(Var/k)$, where $\mathbf{1} = [\operatorname{Spec} k]$?

In [6, 3.3], F. Brown and O. Schnetz conjectured that Question 1.1 is always true for C_1-fields. Question 1.1 over an arbitrary field k is due to H. Esnault in general for the relationship between rational points and the Grothendieck ring $K_0(Var/k)$ (cf. [5, Ques. 3.7]). In [15] Question 1.1 is formulated over algebraically closed fields of characteristic 0 under the name geometric Chevalley–Warning conjecture. Some cases, where Question 1.1 has an affirmative answer for singular hypersurfaces, were worked out in [5] and [15]. Using Brauer group, which yields a well-defined invariant on $K_0(Var/k)/\mathbf{L}$, we give a counter-example to the conjecture of Brown and Schnetz over non-algebraically closed C_1-fields.

Theorem 1.2. *Let X be a smooth projective geometrically integral variety over a field k of characteristic 0. If $[X] \equiv \mathbf{1} \bmod \mathbf{L}$, then $Br(X) \cong Br(k)$.*

The proof of Theorem 1.2 is simple. By Kollár–Larsen–Lunts theorem (cf. [13,14]), one has $[X] \equiv \mathbf{1} \bmod \mathbf{L}$ iff X is stably k-rational. The fact that the Brauer group $Br(X)$ is a birational invariant is due to Grothendieck [12, Cor. 7.3, p. 138]. Moreover, one has $Br(\mathbb{P}^n_X) \cong Br(X)$, because $Br(X)$ can be identified with the unramified Brauer group $Br_{nr}(k(X))$ from the exact sequence (cf. [7, (3.9)])

$$0 \to Br(X) \to Br\big(k(X)\big) \to \bigoplus_{x \in X^{(1)}} H^1_{ét}\big(\kappa(x), \mathbb{Q}/\mathbb{Z}\big)$$

and the latter group $Br_{nr}(k(X))$ gives us a stably birational invariance [8]. So the theorem follows, since $Br(\mathbb{P}^n_k) \cong Br(k)$. In fact, Theorem 1.2 is a special case of a more general invariant coming from strictly $\mathbb{A}^1$-invariant sheaves (see Theorem 1.4 below). However, it is enough to produce a counter-example to the geometric Chevalley–Warning conjecture over non-algebraically closed C_1-fields.

Corollary 1.3. *Let k be a non-algebraically closed field of $char(k) \neq 3$ and assume $k^\times \setminus (k^\times)^3$ is not empty. Let X be a smooth cubic surface given by the equation*

$$x_0^3 + x_1^3 + x_2^3 + a x_3^3 = 0,$$

where $a \notin (k^\times)^3$. Then $Br(X)/Br(k)$ is non-trivial. In particular, if k is a non-algebraically closed C_1-field of characteristic 0 with $k^\times \setminus (k^\times)^3 \neq \emptyset$, then $[X]$ is not $\equiv \mathbf{1} \bmod \mathbf{L}$.

Proof. Obviously $X(k) \neq \emptyset$. If k is a non-algebraically closed field with $char(k) \neq 3$ containing a primitive cubic root of unity, then for the smooth cubic surface as above one has $Br(X)/Br(k) = \mathbb{Z}/3 \oplus \mathbb{Z}/3$ (cf. [16, Ex. 45.3] for number fields and [9, 2.5.1] in general). If k has no primitive cubic roots of unity, the quotient $Br(X)/Br(k)$ is still non-trivial and it is described in [11, Prop. 2.1]. This gives a negative answer to Question 1.1 as desired. □

Now let k be an arbitrary field and let $\mathbf{Ho}_{\mathbb{A}^1}(k)$ be the $\mathbb{A}^1$-homotopy category constructed in [19]. For a space $\mathcal{X} \in \Delta^{op} Sh_{Nis}(Sm/k)$ let $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ be the sheaf associated to the presheaf

$$U \mapsto [U, \mathcal{X}]_{\mathbb{A}^1} \overset{def}{=} \operatorname{Hom}_{\mathbf{Ho}_{\mathbb{A}^1}(k)}(U, \mathcal{X}),$$

for $U \in Sm/k$. We say $\mathcal{X}$ is $\mathbb{A}^1$-connected, if the canonical map $\mathcal{X} \to \operatorname{Spec} k$ induces an isomorphism of sheaves $\pi_0^{\mathbb{A}^1}(\mathcal{X}) \xrightarrow{\simeq} \pi_0^{\mathbb{A}^1}(\operatorname{Spec} k) = \operatorname{Spec} k$, [2]. Let $D_{\mathbb{A}^1}(k)$ denote the $\mathbb{A}^1$-derived category introduced by F. Morel (see e.g. [18, §5.2]). Let us denote by $\mathcal{A}b_k^{\mathbb{A}^1}$ the category of strictly $\mathbb{A}^1$-invariant sheaves (cf. [18, Def. 7, page 8] or [2, Def. 4.3.1]), it is known that $D_{\mathbb{A}^1}(k)$ has a homological t-structure and one can identify $\mathcal{A}b_k^{\mathbb{A}^1}$ with the heart of this t-structure [17, Lem. 6.2.11]. Thus $\mathcal{A}b_k^{\mathbb{A}^1}$ is an abelian category by [4, Thm. 1.3.6]. For a strictly $\mathbb{A}^1$-invariant sheaf M and an irreducible smooth k-scheme X we write $M^{nr}(X)$ for the group of unramified elements [1, Def. 4.1]. Now in the context of $\mathbb{A}^1$-derived category one can prove

Theorem 1.4. *Let k be a field of characteristic 0. If X, Y are two irreducible smooth projective k-varieties, such that $[X] = [Y]$ in $K_0(Var/k)/\mathbf{L}$, then $M(X) \cong M(Y)$ for any strictly $\mathbb{A}^1$-invariant sheaf $M \in \mathcal{A}b_k^{\mathbb{A}^1}$, i.e. M yields a well-defined invariant on $K_0(Var/k)/\mathbf{L}$. In particular, if X is an integral smooth projective k-variety, whose class in $K_0(Var/k)$ satisfies $[X] \equiv \mathbf{1} \bmod \mathbf{L}$, then X is $\mathbb{A}^1$-connected,* *hence for any strictly $\mathbb{A}^1$-invariant sheaf $M \in \mathcal{A}b_k^{\mathbb{A}^1}$ the canonical map $M(k) \to M^{nr}(X)$ is then a bijection, where $M^{nr}(X)$ denotes the group of unramified elements.*

Remark 1.5. Theorem 1.4 is just a simple application of [1, Thm. 3.9]. Our example 1.3 shows that this smooth cubic surface is $\mathbb{A}^1$-disconnected over non-algebraically closed fields, while [2, Cor. 2.4.7] asserts that a smooth proper surface over an algebraically closed field of characteristic 0 is $\mathbb{A}^1$-connected if and only if it is rational.

2. Proof of Theorem 1.4

By Kollár–Larsen–Lunts theorem (cf. [13,14]), one has an isomorphism

$$K_0(Var/k)/\mathbf{L} \to \mathbb{Z}[SB],$$

where the right-hand side denotes the free abelian group generated over the set of stably birational equivalences of smooth projective varieties. So if $[X] = [Y]$ in $K_0(Var/k)/\mathbf{L}$, then X is stably k-birational to Y. We have then $\mathbf{H}_0^{\mathbb{A}^1}(X) \cong \mathbf{H}_0^{\mathbb{A}^1}(Y)$ by [1, Thm. 3.9]. By representing theorem [1, Lem. 3.3], which asserts that

$$H^0_{Nis}(X, M) = \mathrm{Hom}_{\mathcal{A}b_k^{\mathbb{A}^1}}\big(\mathbf{H}_0^{\mathbb{A}^1}(X), M\big),$$

one obtains $M(X) \cong M(Y)$. Remark that one has $M(X) = M^{nr}(X)$, if X is an irreducible smooth k-scheme [1, Lem. 4.2]. Now if X is an integral smooth projective k-variety with $[X] \equiv \mathbf{1} \bmod \mathbf{L}$ in $K_0(Var/k)$, then X is stably k-rational. From [10, Prop. 1.4] one knows that X is then retract k-rational in the sense of Saltman. By [2, Thm. 2.3.6] X is $\mathbb{A}^1$-chain connected, hence $\mathbb{A}^1$-connected by [17, Lem. 6.1.3]. Thus the theorem is proved and we see also immediately that Theorem 1.2 is a special case of Theorem 1.4 by [2, Prop. 4.3.8].

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