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Analytic Geometry Semistability of invariant bundles over G/Γ , II

Semi-stabilité des fibrés vectoriels invariants sur G/Γ , II

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ABSTRACT

Let *G* be a connected complex Lie group, and let Γ be a cocompact discrete subgroup of *G*. We prove that any invariant principal bundle on G/Γ is semistable with respect to any Hermitian structure on G/Γ given by some right-translation invariant Hermitian structure on *G*.

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RÉSUMÉ

Soit *G* un groupe de Lie connexe sur \mathbb{C} , et soit $\Gamma \subset G$ un sous-groupe discret cocompact. Nous démontrons que tout fibré vectoriel invariant sur G/Γ est semi-stable par rapport à toute structure hermitienne sur G/Γ provenant d'une structure hermitienne sur *G* invariante par translations à droite.

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1. Introduction

Let *G* be a complex Lie group, and let $\Gamma \subset G$ be a cocompact discrete subgroup. A holomorphic vector bundle *E* over G/Γ is called invariant if for every $g \in G$, the pullback of *E* by the automorphism of G/Γ defined by $z \mapsto gz$ is holomorphically isomorphic to *E*. In [1] we proved the following under the assumption that *G* is a complex reductive affine algebraic group:

- (1) Any Hermitian form $\tilde{\omega}$ on G/Γ given by right-translations of a *K*-invariant Hermitian structure on Lie(*G*), where *K* is a maximal compact subgroup of *G*, satisfies the identity $d\tilde{\omega}^{n-1} = 0$, where $n = \dim_{\mathbb{C}} G$.
- (2) The degree of any invariant vector bundle over G/Γ is zero.
- (3) Any invariant vector bundle over G/Γ is semistable with respect to the above Hermitian form $\tilde{\omega}$.

Our aim here is to address the general case.

Take any arbitrary pair (G, Γ) , where Γ is a cocompact discrete subgroup of a complex connected Lie group *G*. Fix a Hermitian form ω_0 on Lie(*G*). Let $\tilde{\omega}$ be the Hermitian form on G/Γ defined by the right-translations of ω_0 .

We prove the following (see Corollary 2.2, Theorem 3.1 and Lemma 3.2):

Theorem 1.1.

(1) $d\widetilde{\omega}^{n-1} = 0$, where $n = \dim_{\mathbb{C}} G$.

(2) The degree of any invariant vector bundle over G/Γ is zero.

(3) Any invariant vector bundle over G/Γ is semistable with respect to the above Hermitian form $\tilde{\omega}$.

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Once the first statement in Theorem 1.1 is proved, the proofs of the other two statements are identical to those for the special case of reductive groups dealt in [1].

2. Invariant forms and coclosedness

Let *G* be a connected complex Lie group. The Lie algebra of *G* will be denoted by \mathfrak{g} . For any $g \in G$, let L_g (respectively, R_g) be the left (respectively, right) translation of G by g defined by

 $x \mapsto gx$ (respectively, $x \mapsto xg$).

Fix a Hermitian inner product H_0 on g. Let H be the right-translation invariant Hermitian structure on G obtained by translating H_0 . So H is the unique Hermitian structure on G such that $R^*_{\sigma}H = H$ for all $g \in G$, and $H_e = H_0$, where $e \in G$ is the identity element. This *H* is clearly C^{∞} .

Let ω be the (1, 1)-form on *G* associated to *H*.

Proposition 2.1. Assume that there is a discrete subgroup $\Gamma \subset G$ with the property that the quotient G/Γ is compact. Then

 $d^*\omega = 0.$

where d* is the adjoint of the de Rham differential d with respect to H.

Proof. We will first show that G is unimodular, meaning left-translation invariant Haar measures on G coincide with the right-translation invariant Haar measures.

From Lemma 1.5 of [3, p. 20] it can be deduced that G is unimodular. The details of the argument are as follows: Since Γ in the statement of the proposition is discrete, the modular function Δ_{Γ} on Γ (see [3, p. 17] for its definition) is the constant function 1. Hence the function $(\Delta_G|_{\Gamma})/\Delta_{\Gamma}$ on Γ coincides with Δ_G , where Δ_G is the modular function on G(defined in [3, p. 17]). Therefore, from the criterion in Lemma 1.4 of [3, p. 18] it follows immediately that measure on G/Γ given by a right-translation invariant Haar measure on G is semi-invariant. Now Lemma 1.5 of [3, p. 20] says that G is unimodular.

The tangent bundle of the real manifold *G* will be denoted by $T^{\mathbb{R}}G$. Let

$$J: T^{\mathbb{R}}G \to T^{\mathbb{R}}G$$

be the almost complex structure of the complex manifold G. The Levi-Civita connection on $T^{\mathbb{R}}G$ for the Hermitian structure *H* will be denoted by ∇^{H} . For any $v \in \mathfrak{g}$, let \tilde{v} be the unique right-translation invariant vector field on *G* (section of $T^{\mathbb{R}}G$) such that

$$\widetilde{v}(e) = v.$$

For any $v \in \mathfrak{g}$, let

$$\mathrm{ad}_{\nu}:\mathfrak{g}\to\mathfrak{g}$$

be the derivation defined by $x \mapsto [x, v]$.

With the above notation, it is straightforward to check that

$$\left(\nabla_{\widetilde{u}}^{H}\omega\right)(\widetilde{v},\widetilde{w}) = H\left(\operatorname{ad}_{J(w)}(v),\widetilde{u}\right) \tag{1}$$

for all $u, v, w \in \mathfrak{g}$.

Let n be the complex dimension of g. Fix an orthonormal basis

 $\{e_1, e_2, \ldots, e_{2n}\} \subset \mathfrak{g}$

of the real vector space g for the Hermitian structure H_0 . From (1) we conclude that

$$(\mathbf{d}^*\omega)(\widetilde{u}) = -\sum_{i=1}^{2n} H(\widetilde{\mathrm{ad}}_{J(u)}(e_i), \widetilde{e}_i) = -\mathrm{trace}(\mathrm{ad}_{J(u)});$$
(2)

note that the fact that $(d^*\omega)(\widetilde{u})$ is a constant function follows directly because it is right translation invariant. For any $g \in G$, let

$$\operatorname{Ad}_{\sigma}: G \to G$$

be the automorphism defined by $z \mapsto g^{-1}zg$. Fix a right translation invariant Haar measure μ on *G*. Note that

$$(L_{\sigma^{-1}} \circ R_g)^*(\mu) = \operatorname{Ad}^*_{\sigma}(\mu)$$

for all $g \in G$, where L_g and R_g are the translations by g defined earlier. We showed earlier that G is unimodular. Therefore, from (3) we have

$$\operatorname{Ad}_{g}^{*}(\mu) = \mu$$

for all $g \in G$. Taking the derivative of this identity, we conclude that

 $trace(ad_v) = 0$

for all $v \in \mathfrak{g}$. Now the proof is completed by the identity in (2). \Box

Corollary 2.2. Let G be as in Proposition 2.1. Then

$$\mathrm{d}\omega^{n-1}=0,$$

where *n* is the complex dimension of *G*.

Proof. Let " \star " be the Hodge-star operator on differential forms corresponding to the Hermitian structure *H*. Since $\star d^*\omega = c \cdot d\omega^{n-1}$, where $c \in \mathbb{C} \setminus \{0\}$, it follows from Proposition 2.1 that $d\omega^{n-1} = 0$. \Box

From Corollary 2.2 we have $\partial \overline{\partial} \omega^{n-1} = 0$, meaning *H* is a Gauduchon metric.

3. Degree of invariant bundles and semistability

Take any G as before. Let

 $\Gamma \subset G$

be a discrete subgroup such that G/Γ is compact. Let \widetilde{H} be the Hermitian structure on G/Γ defined by H (recall that H is right-translation invariant). Let $\widetilde{\omega}$ be the (1, 1)-form on G/Γ associated to \widetilde{H} . To $\widetilde{\omega}$ pulls back to ω on G.

For a coherent analytic sheaf \mathcal{E} on G/Γ , define

degree(
$$\mathcal{E}$$
) := $\int_{G/\Gamma} c_1(\det(\mathcal{E})) \wedge \widetilde{\omega}^{n-1} \in \mathbb{R},$

where det(\mathcal{E}) is the determinant line bundle for \mathcal{E} (see [2, Ch. V, §6] for the definition of determinant bundle), and $c_1(\det(\mathcal{E}))$ is a first Chern form for det(\mathcal{E}). Since

 $\mathrm{d}\widetilde{\omega}^{n-1}=0$

(by Corollary 2.2), and any two first Chern forms for $det(\mathcal{E})$ differ by an exact form, it follows that $degree(\mathcal{E})$ is independent of the choice of the first Chern form for $det(\mathcal{E})$.

For any $g \in G$, let $\beta_g : G/\Gamma \to G/\Gamma$ be the biholomorphism defined by L_g . A vector bundle *E* over G/Γ is called *invariant* if $\beta_g^* E$ is holomorphically isomorphic to *E* for all $g \in G$.

Theorem 3.1. Let *E* be an invariant vector bundle over G/Γ . Then

degree(E) = 0.

Proof. The proof is exactly identical to the proof of Theorem 2.2 of [1]. We refrain from repeating it. \Box

A vector bundle *E* over G/Γ is called *semistable* if

$$\frac{\text{degree}(V)}{\text{rank}(V)} \leqslant \frac{\text{degree}(E)}{\text{rank}(E)}$$

for every coherent analytic subsheaf $V \subset E$ of positive rank.

Lemma 3.2. Let *E* be an invariant holomorphic vector bundle on G/Γ . Then *E* is semistable.

Proof. The proof is exactly identical to the proof of Lemma 2.4 of [1]. \Box

As explained in the paragraph after the proof of Lemma 2.4 in [1], Lemma 3.2 generalizes to the following statement: Any invariant holomorphic principal *H*-bundle over G/Γ is semistable, where *H* is any complex reductive affine algebraic group.

(3)

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