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Differential Geometry/Mathematical Physics

# On the projective Randers metrics

### Sur les métriques de Randers projectives

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#### ABSTRACT

It is proved that a Randers metric  $F = \alpha + \beta$  on a manifold of dimension  $n \ge 3$  is projective if and only if the Lie algebra of projective vector fields p(M, F) has (locally) dimension n(n + 2). This can be regarded as an analogue of the corresponding result in Riemannian geometry.

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#### RÉSUMÉ

On démontre qu'une métrique de Randers  $F = \alpha + \beta$  sur une variété de dimension  $n \ge 3$  est projective si et seulement si l'algèbre de Lie des champs de vecteurs projectifs p(M, F) est (localement) de dimension n(n + 2). Ceci peut être considéré comme un analogue du résultat correspondant en géométrie riemannienne.

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#### 1. Introduction

The projective Finsler metrics are smooth solutions to the historic Hilbert's fourth problem. Unlike the Riemannian metrics, a non-projective Finsler metric may be of constant flag curvature in Finsler geometry; see [2]. This causes a failure in legitimacy of Beltrami's theorem in characterizing the Riemannian metrics of constant sectional curvature, see [4] for intuition. This controversial fact is also responsible for concerns regarding the accuracy of other local characterizations of projective Riemannian metrics in Finsler geometry. A celebrated characterization of projective Riemannian metrics deals with the (local) dimension of the Lie algebra of projective vector fields  $p(M, \alpha)$  and presents the maximum projective symmetry in physical terms: a Riemannian metric on a manifold of dimension  $n \ge 3$  is projective if and only if  $\dim(p(M, \alpha)) = n(n+2)$ . The Randers metrics are the most popular Finsler metrics in Differential geometry and Physics simply obtained by a Riemannian metric  $\alpha = \sqrt{a_{ij}(x)y^iy^j}$  and a 1-form  $\beta = b_i(x)y^i$  as  $F = \alpha + \beta$  and were introduced by G. Randers in [10] in the contexts of General Relativity. Nevertheless, the projective Randers metrics of isotropic S-curvature are locally characterized by Chen et al. in [3]. Moreover, the projective Randers metrics of constant S-curvature are locally characterized in [7,8]. Here, we establish the following characterization of projective Randers metrics:

**Theorem 1.1.** A Randers metric  $F = \alpha + \beta$  on a manifold M of dimension  $(n \ge 3)$  is projective if and only if p(M, F) has (locally) dimension n(n + 2).

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The horizontal and vertical derivations are exerted with respect to the Berwald connection and are denoted by subscripts |i| and  $j_i$ , respectively. Moreover, we deal with pure Randers metrics, i.e.  $\beta \neq 0$ .

#### 2. Projective vector fields on Randers spaces

Every vector field X on M induces naturally a transformation under the following infinitesimal coordinate transformations on *TM*,  $(x^i, y^i) \rightarrow (\bar{x}^i, \bar{y}^i)$  given by  $\bar{x}^i = x^i + V^i dt$ ,  $\bar{y}^i = y^i + y^k \frac{\partial V^i}{\partial x^k} dt$ . This leads us to the notion of *the complete lift*  $\hat{V}$  of V to a vector field on *TM*<sub>0</sub> given by

$$\hat{V} = V^i \frac{\partial}{\partial x^i} + y^k \frac{\partial V^i}{\partial x^k} \frac{\partial}{\partial y^i}.$$

The Lie derivatives of Finslerian geometric objects should be regarded with respect to  $\hat{V}$ . Notice that,  $\pounds_{\hat{V}} y^i = 0$ ,  $\pounds_{\hat{V}} dx^i = 0$ and the differential operators  $\pounds_{\hat{V}}$ ,  $\frac{\partial}{\partial x^i}$ , exterior differential operator d and  $\frac{\partial}{\partial y^i}$  commute. The vector field V is called a projective vector field, if there is a function P on  $TM_0$  such that  $\pounds_{\hat{V}} G^i{}_k = P \delta^i{}_k + P_k y^i$ , where  $P_k = P_{.k}$ , see [1]. In this case, given any appropriate t, the local flow  $\{\phi_t\}$  associated to V is projective transformation. If V is a projective vector field, then [1,6]:

$$\begin{split} & \pounds_{\hat{V}} G^{i} = P y^{i}, \\ & \pounds_{\hat{V}} G^{i}{}_{jk} = \delta^{i}{}_{j} P_{k} + \delta^{i}{}_{k} P_{j} + y^{i} P_{k.j}, \\ & \pounds_{\hat{V}} G^{i}{}_{jkl} = \delta^{i}{}_{j} P_{k.l} + \delta^{i}{}_{l} P_{j.l} + \delta^{i}{}_{l} P_{k.j} + y^{i} P_{k.j.l}, \\ & \pounds_{\hat{V}} G_{jl} = (n+1) P_{j.l}, \end{split}$$
(1)

where,  $G^{i}_{j} = G^{i}_{j,j}$ ,  $G^{i}_{jk} = G^{i}_{j,k}$ ,  $G^{i}_{jkl} = G^{i}_{jk,l}$  and  $G_{jl} = G^{i}_{jil}$ . A projective vector field is called *affine* if P = 0. Every Killing vector field is affine. On the Riemannian spaces, given any projective vector field V the function P = P(x, y) is linear with respect to y, while in the Finslerian setting the mentioned linearity is a non-Riemannian obstruction. A projective vector field V is called a *special projective vector field* if  $\pounds_{\hat{V}} G_{jl} = 0$ , equivalently,  $P(x, y) = P_i(x)y^i$  due to (1).

Let  $(M, \alpha)$  be a Riemannian space and  $\beta = b_i(x)y^i$  be a 1-form defined on M such that  $\|\beta\|_x := \sup_{y \in T_xM} \beta(y)/\alpha(y) < 1$ . The Finsler metric  $F = \alpha + \beta$  is called a Randers metric on a manifold M. Denote the geodesic spray coefficients of  $\alpha$  and F by the notions  $G^i_{\alpha}$  and  $G^i$ , respectively and the Levi-Civita connection of  $\alpha$  by  $\nabla$ . Define  $\nabla_j b_i$  by  $(\nabla_j b_i)\theta^j := db_i - b_j\theta_i^{j}$ , where  $\theta^i := dx^i$  and  $\theta_i^{j} := \tilde{\Gamma}^{ij}_{ik} dx^k$  denote the Levi-Civita connection forms and  $\nabla$  denotes its associated covariant derivation of  $\alpha$ . Let us put

$$r_{ij} := \frac{1}{2} (\nabla_j b_i + \nabla_i b_j), \qquad s_{ij} := \frac{1}{2} (\nabla_j b_i - \nabla_i b_j),$$
  

$$s^i{}_j := a^{ih} s_{hj}, \qquad s_j := b_i s^i{}_j, \qquad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

Then  $G^i$  are given by

$$G^{i} = G^{i}_{\alpha} + \left(\frac{e_{\circ\circ}}{2F} - s_{\circ}\right)y^{i} + \alpha s^{i}_{\circ},$$
<sup>(2)</sup>

where  $e_{\infty} := e_{ij}y^i y^j$ ,  $s_{\infty} := s_i y^i$ ,  $s_{\infty}^i := s_i^i y^j$  and  $G_{\alpha}^i$  denote the geodesic spray coefficients of  $\alpha$ , see [11].

The projective vector fields are variously characterized in many contexts such as [1]. The projective vector fields in a Randers space  $(M, F = \alpha + \beta)$  can be characterized in terms of  $\alpha$  and  $\beta$  in the following theorem:

**Theorem 2.1.** (See [7,9].) A vector field V is projective on a Randers space  $(M, F = \alpha + \beta)$  if and only if V is projective in  $(M, \alpha)$  and  $\pounds_{\hat{V}}(\alpha s^i_j) = 0$ .

**Remark 1.** Theorem 2.1 follows that the Lie algebra of projective vector fields in (M, F) is a Lie sub-algebra of the Lie algebra of projective vector fields in  $(M, \alpha)$ , namely  $p(M, F) \subseteq p(M, \alpha)$ . Hence, we have the inequalities  $\dim(p(M, F)) \leq \dim(p(M, \alpha)) \leq n(n+2)$ .

#### 3. Proof of main theorem

Suppose that we have  $\dim(p(M, F)) = n(n + 2)$ . By Remark 1, it follows that  $\dim(p(M, \alpha)) = n(n + 2)$  as well as  $p(M, F) = p(M, \alpha)$ . This results that  $\alpha$  is of maximum projective symmetry and thus it is of constant sectional curvature, say k. Moreover, every Killing vector field V in  $(M, \alpha)$  is projective vector field in (M, F). It is also well known that the Killing vector fields V are locally of the form

$$V^{i} = Q^{i}_{k} x^{k} + C^{i} + k \langle x, C \rangle x^{i},$$
(3)

where, *C* is an arbitrary constant vector and  $Q_k^i$  is an arbitrary constant skew-symmetry matrix. On the other hand, by Theorem 2.1,  $f_{\hat{V}}(\alpha s_j^i) = 0$ . *V* is a Killing vector field and hence  $f_{\hat{V}}s_{ij} = f_V s_{ij} = 0$ . This provides the following equation:

$$\pounds_V s_{ij} = \frac{\partial V^k}{\partial x^j} s_{ik} + \frac{\partial V^k}{\partial x^i} s_{kj} + V^k \frac{\partial}{\partial x^k} s_{ij} = 0.$$
(4)

Let us assume C = 0 in the sequel. From (3), we obtain

$$\frac{\partial V^{k}}{\partial x^{j}} = Q^{k}{}_{j}, \qquad \frac{\partial V^{k}}{\partial x^{i}} = Q^{k}{}_{i}.$$
(5)

Plugging the terms  $\frac{\partial V^k}{\partial x^j}$  and  $\frac{\partial V^k}{\partial x^i}$  from (5) in (4), we infer:

$$Q^{k}{}_{j}s_{ik} + Q^{k}{}_{i}s_{kj} + Q^{k}{}_{l}x^{l}\frac{\partial}{\partial x^{k}}s_{ij} = 0,$$
(6)

where,  $Q = (Q^{k}_{j})$  is an arbitrary skew-symmetric matrix. Consider two fixed distinct indices  $l_{0}$  and  $k_{0}$  such that  $Q^{k_{0}}_{l_{0}} = -Q^{l_{0}}_{k_{0}} = 1$  and  $Q^{k}_{l} = 0$  if  $k \neq k_{0}$  or  $l \neq l_{0}$ . Given any indices i and j such that  $i, j \neq l_{0}$ , we have

$$Q^{k}{}_{j}s_{ik} = 0, \qquad Q^{k}{}_{i}s_{kj} = 0, \qquad Q^{k}{}_{l}x^{l} = \begin{cases} x^{l_{0}}, & k = k_{0}, \\ -x^{k_{0}}, & k = l_{0}, \\ 0, & \text{otherwise} \end{cases}$$

(6) becomes  $(x^{l_0} - x^{k_0}) \cdot \frac{\partial}{\partial x^{k_0}} s_{ij} = 0$ . It follows that,  $\frac{\partial}{\partial x^k} s_{ij} = 0$  if  $i, j \neq k$ . Now, fix two distinct indices i and j and consider the matrix Q given by  $Q^i_{\ j} = -Q^j_{\ i} = 1$  and  $Q^k_{\ l} = 0$  if  $k \neq i$  or  $l \neq l_0$ . Observe that for the matrix Q we have

$$Q^{k}{}_{j}s_{ik} = Q^{i}{}_{j}s_{ii} = 0, \qquad Q^{k}{}_{i}s_{kj} = Q^{j}{}_{i}s_{jj} = 0, \qquad Q^{k}{}_{l}x^{l} = \begin{cases} x^{i}, & k = j, \\ -x^{j}, & k = i, \\ 0, & \text{otherwise}, \end{cases}$$

and (6) becomes  $(x^i - x^j) \cdot \frac{\partial}{\partial x^j} s_{ij} = 0$ . It follows then given any two indices *i* and *j*, we have  $\frac{\partial}{\partial x^j} s_{ij} = 0$ . Finally, it results that given any three indices *i*, *j* and *k*, we have  $\frac{\partial}{\partial x^k} s_{ij} = 0$ . Plugging this in (6) we obtain  $Q^k_{\ i} s_{ik} + Q^k_{\ i} s_{kj} = 0$ . Now, let  $i \neq j$  and  $k_0 \neq i$ , *j* and  $Q^{k_0}_i = -Q^i_{\ k_0} = 1$  and  $Q^k_l = 0$  if  $k \neq k_0$  or  $l \neq i$ . Thus, (6) can be written as follows:  $Q^k_{\ i} s_{ik} + Q^k_{\ i} s_{kj} = s_{k_0 j} = 0$ . Since *i* and *j* are arbitrarily chosen, hence  $s_{ij} = 0$ . Namely, the 1-form  $\beta$  is closed. But,  $\alpha$  has been already proved to have constant sectional curvature. Summarizing,  $\beta$  is closed,  $\alpha$  is constant sectional curvature. This is exactly when  $F = \alpha + \beta$  is locally projectively flat and has scalar flag curvature.

Conversely, let us suppose that  $F = \alpha + \beta$  is projective, equivalently  $\alpha$  has constant sectional curvature and  $\beta$  is closed; in particular, dim $(p(M, \alpha)) = n(n + 2)$ . The 1-form  $\beta$  is closed, hence  $s_{ij} = 0$  and by (2) F and  $\alpha$  are projectively related. In this case, we have  $p(M, \alpha) = p(M, F)$  and we obtain dim $(p(M, F)) = \dim(p(M, \alpha)) = n(n + 2)$ .  $\Box$ 

**Remark 2.** The very recent work [5] shows that there are topological obstructions for a projective Randers space to have an n(n + 2)-dimensional Lie algebra of projective vector fields.

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