## Differential Geometry/Mathematical Physics

# On the projective Randers metrics 

## Sur les métriques de Randers projectives

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## ARTICLE INFO

## Article history:

Received 6 November 2011
Accepted 28 February 2012
Available online 16 March 2012
Presented by the Editorial Board


#### Abstract

It is proved that a Randers metric $F=\alpha+\beta$ on a manifold of dimension $n \geqslant 3$ is projective if and only if the Lie algebra of projective vector fields $p(M, F)$ has (locally) dimension $n(n+2)$. This can be regarded as an analogue of the corresponding result in Riemannian geometry. © 2012 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. R É S U M É

On démontre qu'une métrique de Randers $F=\alpha+\beta$ sur une variété de dimension $n \geqslant 3$ est projective si et seulement si l'algèbre de Lie des champs de vecteurs projectifs $p(M, F)$ est (localement) de dimension $n(n+2)$. Ceci peut être considéré comme un analogue du résultat correspondant en géométrie riemannienne.


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## 1. Introduction

The projective Finsler metrics are smooth solutions to the historic Hilbert's fourth problem. Unlike the Riemannian metrics, a non-projective Finsler metric may be of constant flag curvature in Finsler geometry; see [2]. This causes a failure in legitimacy of Beltrami's theorem in characterizing the Riemannian metrics of constant sectional curvature, see [4] for intuition. This controversial fact is also responsible for concerns regarding the accuracy of other local characterizations of projective Riemannian metrics in Finsler geometry. A celebrated characterization of projective Riemannian metrics deals with the (local) dimension of the Lie algebra of projective vector fields $p(M, \alpha)$ and presents the maximum projective symmetry in physical terms: a Riemannian metric on a manifold of dimension $n \geqslant 3$ is projective if and only if $\operatorname{dim}(p(M, \alpha))=$ $n(n+2)$. The Randers metrics are the most popular Finsler metrics in Differential geometry and Physics simply obtained by a Riemannian metric $\alpha=\sqrt{a_{i j}(x) y^{i} y^{j}}$ and a 1-form $\beta=b_{i}(x) y^{i}$ as $F=\alpha+\beta$ and were introduced by G. Randers in [10] in the contexts of General Relativity. Nevertheless, the projective Randers metrics of isotropic S-curvature are locally characterized by Chen et al. in [3]. Moreover, the projective Randers metrics of constant S-curvature are locally characterized in [7,8]. Here, we establish the following characterization of projective Randers metrics:

Theorem 1.1. A Randers metric $F=\alpha+\beta$ on a manifold $M$ of dimension $(n \geqslant 3)$ is projective if and only if $p(M, F)$ has (locally) dimension $n(n+2)$.

[^0]The horizontal and vertical derivations are exerted with respect to the Berwald connection and are denoted by subscripts $\mid i$ and ${ }_{i}$, respectively. Moreover, we deal with pure Randers metrics, i.e. $\beta \neq 0$.

## 2. Projective vector fields on Randers spaces

Every vector field $X$ on $M$ induces naturally a transformation under the following infinitesimal coordinate transformations on $T M,\left(x^{i}, y^{i}\right) \rightarrow\left(\bar{x}^{i}, \bar{y}^{i}\right)$ given by $\bar{x}^{i}=x^{i}+V^{i} \mathrm{~d} t, \bar{y}^{i}=y^{i}+y^{k} \frac{\partial V^{i}}{\partial x^{k}} \mathrm{~d} t$. This leads us to the notion of the complete lift $\hat{V}$ of $V$ to a vector field on $T M_{0}$ given by

$$
\hat{V}=V^{i} \frac{\partial}{\partial x^{i}}+y^{k} \frac{\partial V^{i}}{\partial x^{k}} \frac{\partial}{\partial y^{i}}
$$

The Lie derivatives of Finslerian geometric objects should be regarded with respect to $\hat{V}$. Notice that, $£_{\hat{V}} y^{i}=0, £_{\hat{V}} \mathrm{~d} x^{i}=0$ and the differential operators $£_{\hat{V}}, \frac{\partial}{\partial x^{i}}$, exterior differential operator d and $\frac{\partial}{\partial y^{i}}$ commute. The vector field $V$ is called a projective vector field, if there is a function $P$ on $T M_{0}$ such that $£_{\hat{V}} G^{i}{ }_{k}=P \delta^{i}{ }_{k}+P_{k} y^{i}$, where $P_{k}=P_{. k}$, see [1]. In this case, given any appropriate $t$, the local flow $\left\{\phi_{t}\right\}$ associated to $V$ is projective transformation. If $V$ is a projective vector field, then [1,6]:

$$
\begin{align*}
& £_{\hat{V}} G^{i}=P y^{i}, \\
& £_{\hat{V}} G^{i}{ }_{j k}=\delta^{i}{ }_{j} P_{k}+\delta^{i}{ }_{k} P_{j}+y^{i} P_{k . j}, \\
& £_{\hat{V}} G^{i}{ }_{j k l}=\delta^{i}{ }_{j} P_{k . l}+\delta^{i}{ }_{k} P_{j . l}+\delta^{i}{ }_{l} P_{k . j}+y^{i} P_{k . j . l}, \\
& £_{\hat{V}} G_{j l}=(n+1) P_{j . l}, \tag{1}
\end{align*}
$$

where, $G^{i}{ }_{j}=G_{. j}^{i}, G^{i}{ }_{j k}=G_{j . k}^{i}, G^{i}{ }_{j k l}=G_{j k . l}^{i}$ and $G_{j l}=G^{i}{ }_{j i l}$. A projective vector field is called affine if $P=0$. Every Killing vector field is affine. On the Riemannian spaces, given any projective vector field $V$ the function $P=P(x, y)$ is linear with respect to $y$, while in the Finslerian setting the mentioned linearity is a non-Riemannian obstruction. A projective vector field $V$ is called a special projective vector field if $£_{\hat{V}} G_{j l}=0$, equivalently, $P(x, y)=P_{i}(x) y^{i}$ due to (1).

Let $(M, \alpha)$ be a Riemannian space and $\beta=b_{i}(x) y^{i}$ be a 1 -form defined on $M$ such that $\|\beta\|_{x}:=\sup _{y \in T_{x} M} \beta(y) / \alpha(y)<1$. The Finsler metric $F=\alpha+\beta$ is called a Randers metric on a manifold $M$. Denote the geodesic spray coefficients of $\alpha$ and $F$ by the notions $G_{\alpha}^{i}$ and $G^{i}$, respectively and the Levi-Civita connection of $\alpha$ by $\nabla$. Define $\nabla_{j} b_{i}$ by $\left(\nabla_{j} b_{i}\right) \theta^{j}:=\mathrm{d} b_{i}-b_{j} \theta_{i}{ }^{j}$, where $\theta^{i}:=\mathrm{d} x^{i}$ and $\theta_{i}{ }^{j}:=\tilde{\Gamma}_{i k}^{j} \mathrm{~d} x^{k}$ denote the Levi-Civita connection forms and $\nabla$ denotes its associated covariant derivation of $\alpha$. Let us put

$$
\begin{aligned}
& r_{i j}:=\frac{1}{2}\left(\nabla_{j} b_{i}+\nabla_{i} b_{j}\right), \quad s_{i j}:=\frac{1}{2}\left(\nabla_{j} b_{i}-\nabla_{i} b_{j}\right), \\
& s_{j}^{i}:=a^{i h} s_{h j}, \quad s_{j}:=b_{i} s_{j}^{i}, \quad e_{i j}:=r_{i j}+b_{i} s_{j}+b_{j} s_{i} .
\end{aligned}
$$

Then $G^{i}$ are given by

$$
\begin{equation*}
G^{i}=G_{\alpha}^{i}+\left(\frac{e_{\circ \circ}}{2 F}-s_{\circ}\right) y^{i}+\alpha s^{i}{ }_{\circ} \tag{2}
\end{equation*}
$$

where $e_{\circ \circ}:=e_{i j} y^{i} y^{j}, s_{\circ}:=s_{i} y^{i}, s^{i}{ }_{\circ}:=s^{i}{ }_{j} y^{j}$ and $G_{\alpha}^{i}$ denote the geodesic spray coefficients of $\alpha$, see [11].
The projective vector fields are variously characterized in many contexts such as [1]. The projective vector fields in a Randers space $(M, F=\alpha+\beta)$ can be characterized in terms of $\alpha$ and $\beta$ in the following theorem:

Theorem 2.1. (See $[7,9])$.$A vector field V$ is projective on a Randers space $(M, F=\alpha+\beta)$ if and only if $V$ is projective in $(M, \alpha)$ and $£_{\hat{V}}\left(\alpha s^{i}{ }_{j}\right)=0$.

Remark 1. Theorem 2.1 follows that the Lie algebra of projective vector fields in $(M, F)$ is a Lie sub-algebra of the Lie algebra of projective vector fields in $(M, \alpha)$, namely $p(M, F) \subseteq p(M, \alpha)$. Hence, we have the inequalities $\operatorname{dim}(p(M, F)) \leqslant$ $\operatorname{dim}(p(M, \alpha)) \leqslant n(n+2)$.

## 3. Proof of main theorem

Suppose that we have $\operatorname{dim}(p(M, F))=n(n+2)$. By Remark 1, it follows that $\operatorname{dim}(p(M, \alpha))=n(n+2)$ as well as $p(M, F)=p(M, \alpha)$. This results that $\alpha$ is of maximum projective symmetry and thus it is of constant sectional curvature, say $k$. Moreover, every Killing vector field $V$ in $(M, \alpha)$ is projective vector field in $(M, F)$. It is also well known that the Killing vector fields $V$ are locally of the form

$$
\begin{equation*}
V^{i}=Q^{i}{ }_{k} x^{k}+C^{i}+k\langle x, C\rangle x^{i} \tag{3}
\end{equation*}
$$

where, $C$ is an arbitrary constant vector and $Q^{i}{ }_{k}$ is an arbitrary constant skew-symmetry matrix. On the other hand, by Theorem 2.1, $£_{\hat{V}}\left(\alpha s^{i}{ }_{j}\right)=0 . V$ is a Killing vector field and hence $£_{\hat{V}} s_{i j}=£_{V} s_{i j}=0$. This provides the following equation:

$$
\begin{equation*}
£_{V} s_{i j}=\frac{\partial V^{k}}{\partial x^{j}} s_{i k}+\frac{\partial V^{k}}{\partial x^{i}} s_{k j}+V^{k} \frac{\partial}{\partial x^{k}} s_{i j}=0 . \tag{4}
\end{equation*}
$$

Let us assume $C=0$ in the sequel. From (3), we obtain

$$
\begin{equation*}
\frac{\partial V^{k}}{\partial x^{j}}=Q_{j}^{k}, \quad \frac{\partial V^{k}}{\partial x^{i}}=Q_{i}^{k} \tag{5}
\end{equation*}
$$

Plugging the terms $\frac{\partial V^{k}}{\partial x^{j}}$ and $\frac{\partial V^{k}}{\partial x^{i}}$ from (5) in (4), we infer:

$$
\begin{equation*}
Q^{k}{ }_{j} s_{i k}+Q^{k}{ }_{i} s_{k j}+Q^{k}{ }_{l} x^{l} \frac{\partial}{\partial x^{k}} s_{i j}=0, \tag{6}
\end{equation*}
$$

where, $Q=\left(Q^{k}{ }_{j}\right)$ is an arbitrary skew-symmetric matrix. Consider two fixed distinct indices $l_{0}$ and $k_{0}$ such that $Q^{k_{0}} l_{0}=$ $-Q^{l_{0}}{ }_{k_{0}}=1$ and $Q^{k}{ }_{l}=0$ if $k \neq k_{0}$ or $l \neq l_{0}$. Given any indices $i$ and $j$ such that $i, j \neq l_{0}$, we have

$$
Q^{k}{ }_{j} s_{i k}=0, \quad Q^{k}{ }_{i} s_{k j}=0, \quad Q^{k}{ }_{l} x^{l}= \begin{cases}x^{l_{0}}, & k=k_{0} \\ -x^{k_{0}}, & k=l_{0} \\ 0, & \text { otherwise }\end{cases}
$$

(6) becomes $\left(x^{l_{0}}-x^{k_{0}}\right) \cdot \frac{\partial}{\partial x^{k_{0}}} s_{i j}=0$. It follows that, $\frac{\partial}{\partial x^{k}} s_{i j}=0$ if $i, j \neq k$. Now, fix two distinct indices $i$ and $j$ and consider the matrix $Q$ given by $Q^{i}{ }_{j}=-Q^{j}{ }_{i}=1$ and $Q^{k}{ }_{l}=0$ if $k \neq i$ or $l \neq l_{0}$. Observe that for the matrix $Q$ we have

$$
Q^{k}{ }_{j} s_{i k}=Q^{i}{ }_{j} s_{i i}=0, \quad Q^{k}{ }_{i} s_{k j}=Q^{j}{ }_{i} s_{j j}=0, \quad Q^{k}{ }_{l} x^{l}= \begin{cases}x^{i}, & k=j \\ -x^{j}, & k=i \\ 0, & \text { otherwise }\end{cases}
$$

and (6) becomes $\left(x^{i}-x^{j}\right) \cdot \frac{\partial}{\partial x^{j}} s_{i j}=0$. It follows then given any two indices $i$ and $j$, we have $\frac{\partial}{\partial x^{j}} s_{i j}=0$. Finally, it results that given any three indices $i, j$ and $k$, we have $\frac{\partial}{\partial x^{k}} s_{i j}=0$. Plugging this in (6) we obtain $Q^{k}{ }_{j} s_{i k}+Q^{k}{ }_{i} s_{k j}=0$. Now, let $i \neq j$ and $k_{0} \neq i, j$ and $Q^{k_{0}}{ }_{i}=-Q^{i}{ }_{k_{0}}=1$ and $Q^{k}{ }_{l}=0$ if $k \neq k_{0}$ or $l \neq i$. Thus, (6) can be written as follows: $Q^{k}{ }_{j} s_{i k}+Q^{k}{ }_{i} s_{k j}=s_{k_{0} j}=0$. Since $i$ and $j$ are arbitrarily chosen, hence $s_{i j}=0$. Namely, the 1 -form $\beta$ is closed. But, $\alpha$ has been already proved to have constant sectional curvature. Summarizing, $\beta$ is closed, $\alpha$ is constant sectional curvature. This is exactly when $F=\alpha+\beta$ is locally projectively flat and has scalar flag curvature.

Conversely, let us suppose that $F=\alpha+\beta$ is projective, equivalently $\alpha$ has constant sectional curvature and $\beta$ is closed; in particular, $\operatorname{dim}(p(M, \alpha))=n(n+2)$. The 1-form $\beta$ is closed, hence $s_{i j}=0$ and by (2) $F$ and $\alpha$ are projectively related. In this case, we have $p(M, \alpha)=p(M, F)$ and we obtain $\operatorname{dim}(p(M, F))=\operatorname{dim}(p(M, \alpha))=n(n+2)$.

Remark 2. The very recent work [5] shows that there are topological obstructions for a projective Randers space to have an $n(n+2)$-dimensional Lie algebra of projective vector fields.

## Acknowledgement

The first author would like to express his sincere thanks for this work was in part supported by the Institute for Research in Fundamental Sciences (IPM) under the grant No. 89350036.

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