1. Introduction

For each positive integer $m$, let $K^m$ be the $m$-Kronecker quiver which consists of two vertices and $m$ arrows from one to the other. For generic non-trivial stability conditions [1] on the category of representations of $K^m$ and moduli spaces of stable representations $M(K^m(a, b))$ of coprime dimension vectors $(a, b)$ [5], we study Euler characteristics $\chi(K^m(a, b))$.

We give some more details in the later section and we go on as follows. Notice that for the Euler form $\langle \cdot , \cdot \rangle$ and a symplectic form $\{ \cdot , \cdot \}$, which is an anti-symmetrization of the Euler form, we may take a non-trivial stability condition on the category of representations of $K^m$ such that for representations $E, F$ of $K^m$ and the slope function $\mu$, we have $\mu(E) > \mu(F)$ if and only if $\{ E, F \} > 0$.

For objects to study in terms of wall-crossings, stability conditions such that the positivity of the difference of slopes coincides with that of symplectic forms on the Grothendieck group have been commonly called Denef’s stability conditions in physics [2]. We employ these special stability conditions and the terminology.

Euler characteristics $\chi(K^m(a, b))$ have been studied extensively. In particular, formulas of Kontsevich–Soibelman and Reineke [6,10,12] have been known. In this article, we would like to study quantitative questions for $m \gg 0$.

To analyze further, for each coprime $a, b$ and $m > 0$, let us define the bipartite quiver $Q^m(a, b)$ which consists of $a$ source vertices and $b$ terminal vertices with $m$ arrows from each source vertex to each terminal vertex. On representations of $Q^m(a, b)$, we have Denef’s stability conditions (see Section 2).
We denote $M(Q^m(a, b))$ to be the moduli space of stable representations of dimension vectors being one on each vertex of $Q^m(a, b)$ and $\chi(Q^m(a, b))$ to be the corresponding Euler characteristic. We have the following:

**Theorem 1.** For each coprime $a, b, \text{ and } m \gg 0$, we have

$$\chi(Q^1(a, b)) \sim \frac{a!b!}{m^{a+b-1}} \chi(K^m(a, b)).$$

We would like to mention that in Theorem 1, Euler characteristics in the left-hand and right-hand sides are discussed in terms of blackhole counting in supergravity [7] and Witten index in superstring theory [3] (see also [15]).

Key tools to obtain Theorem 1 are the recently obtained formula in Theorem 3 on $\chi(K^m(a, b))$ by Manschot, Pioline and Sen [7] (MPS formula for short, see also [8,9,14])\(^1\) and our Lemma 2.1. We realize that by taking $m$ and arbitrarily large, we see that Stirling formula explains Theorem 1.

Douglas has conjectured the following [4,11,16]. For coprime $a, b \gg 0$ such that $\frac{b}{a} \sim r$ and each $m$, we have that

$$\ln(\chi(K^m(a, b))) \sim (a + b - 1) \ln(m).$$

In particular, for $a, b \gg 0$ such that $\frac{b}{a} \sim r$ and large enough $m$ depending on $a, b$, we have

$$\frac{\ln(\chi(K^m(a, b)))}{a} \sim (1 + r) \ln(m).$$

2. Proofs

Let us expand and introduce notions. For each $a$, let $\bar{a}$ denote a partition of $a$ such that for non-negative integers $a_l$ of $l \geq 1$, we have $\sum l a_l = a$. We put $S_\bar{a} = \sum a_l$ for our convenience. When $a_1 = a$, we simply write $a$ for $\bar{a}$. For a quiver $Q$ and representations $E, F$ of $Q$, on the Grothendieck group of the category of representations of $Q$, let $(E, F)_Q$ be the Euler form and $(E, F)_Q$ the symplectic form $(F, E)_{Q} \sim -(E, F)_Q$. For a dimension vector $d$, we call a partition $d_1, \ldots, d_s$ of $d$ such that $\sum_{p=1}^s d_p = d$ and $\{\sum_{p=1}^b d_p, d\}_Q > 0$ for each $b = 1, \ldots, s$ to be admissible.

For each $m > 0$ and partitions $\bar{a}, \bar{b}$ of $a$ and $b$, we define the bipartite quiver $Q^m(\bar{a}, \bar{b})$ as follows. It consists of $S_{\bar{a}}$ source vertices such that for each $l$, we have $a_l$ vertices $v_l$ for our convenience, we say $a_l$ is the label of $v_l$ and we put $w(v_l) = l$. It consists of $S_{\bar{b}}$ terminal vertices with labels and $w(-)$ being defined by the same manner. We put $m w(v)w(v')$ arrows from each source vertex $v$ to each terminal vertex $v'$.

Let us explain Denef’s stability conditions in use. For the $m$-Kronecker quiver $Q^m$, the source vertex $(1, 0)$, and the terminal vertex $(0, 1)$, the slope function $\mu$ satisfies $\mu(1, 0) > \mu(0, 1)$. For $Q^m(\bar{a}, \bar{b})$ and vertices $v$ and $v'$ with the labels being $a_l$ and $b_l$, central charges $\frac{Z(v)}{w(v)}$ and $\frac{Z(v')}{w(v')}$ coincide with those of the vertices $(1, 0)$ and $(0, 1)$.

We write $(\bar{a}, \bar{b})$ for the dimension vector which has one on each vertex of the quiver $Q^m(\bar{a}, \bar{b})$. We let $M(Q^m(\bar{a}, \bar{b}))$ be the moduli space of stable representations of the dimension vector $(\bar{a}, \bar{b})$ of $Q^m(\bar{a}, \bar{b})$. We denote $P(Q^m(\bar{a}, \bar{b}), y)$ to be the Poincaré polynomial and we put $\chi(Q^m(\bar{a}, \bar{b})) = P(Q^m(\bar{a}, \bar{b}), 1)$. For the $m$-Kronecker quiver $K^m$, we have the following MPS formula by specializing the Poincaré polynomial formula in [7, Appendix D]:

**Theorem 3 (MPS formula).** For each coprime $a, b, \text{ and } m > 0$, we have

$$\chi(K^m(a, b)) = \sum_{\bar{a}, \bar{b}} \chi(Q^m(\bar{a}, \bar{b})) \cdot \prod_{l} \frac{1}{a_l!} \left( \frac{(-1)^{b_l} l!}{l^{2b_l}} \right) \cdot \prod_{l} \frac{1}{b_l!} \left( \frac{(-1)^{a_l} l!}{l^{2a_l}} \right).$$

Notice that $M(Q^m(\bar{a}, \bar{b}))$ is a non-trivial smooth projective variety, since we have stable representations including ones with invertible maps on every arrow. We have the following:

\(^1\) In [7], they give their formula in terms of Poincaré polynomials for Denef’s stabilities on quivers without oriented loops. We use its Euler characteristic version on Kronecker quivers. In [13], their formula has been motivically generalized and, for complete bipartite quivers and Euler characteristics, identified with a degeneration formula of Gromow–Witten theory.
Lemma 2.1.
\[ \chi(Q^m(\vec{a}, \vec{b})) = m^{5z + 3z - 1} \chi(Q^1(\vec{a}, \vec{b})). \]

**Proof.** We consider the Poincaré polynomial \( P(Q^m(\vec{a}, \vec{b}), y) \) with Reineke’s formula [10, Corollary 6.8]. For the dimension vector \((\vec{a}, \vec{b})\), we take an admissible partition \(d^1, \ldots, d^t \) and \((-1)^{t-1}y^2\sum_{i<j}a_i d_i a_j\). We notice that \( \{-\}Q^m(\vec{a}, \vec{b}) = m^1 \cdot Q^1(\vec{a}, \vec{b}) \). The set of admissible partitions is invariant under choices of \(m\). For each admissible partition, the power of \( y \) above is the \( m \) times of that for \( P(Q^1(\vec{a}, \vec{b}), y) \). We have that \( P(Q^1(\vec{a}, \vec{b}), y) \) is a non-zero polynomial. Ignoring an overall factor of a power of \( y \) and writing \( y^2 \) as \( q \) for simplicity, for some non-trivial and non-negative integers \( a_i \) and \( b_i \), we have \( P(Q^1(\vec{a}, \vec{b}), q) = (q - 1)^{5z - 3z} (\sum_{i \geq 0} a_i (q - 1)^{5z + 3z - 1} q^k) \). For admissible partitions, the second factor is the sum of terms above. So we have \( P(Q^m(\vec{a}, \vec{b}), q) = (q - 1)^{5z - 3z} (\sum_{i \geq 0} a_i (q^m - 1)^{5z + 3z - 1} q^m) \). □

We give a proof of Theorem 1.

**Proof.** By Lemma 2.1, \( \chi(Q^m(a, b)) \) carries the highest power of \( m \) among \( \chi(Q^m(\vec{a}, \vec{b})) \) in Theorem 3. □

We give a proof of Corollary 2.

**Proof.** When \( a + b = 1 \), \( M(K^m(a, b)) \) is a point. For \( a + b \neq 1 \) and large enough \( m \) so that
\[
\left| \ln\left( \frac{\chi(Q^1(a, b))}{(a + b) \ln(m)} \right) \right| \ll 1,
\]
the first assertion follows. For the second assertion, with \( a_i, b_i, m_i \) such that \( b_i \rightarrow r, \frac{1}{a_i} \rightarrow 0, \) and \( \frac{\ln(\chi(Q^m(a_i, b_i)))}{\ln(m_i(a_i + b_i - 1))} \rightarrow 1 \) for \( i \rightarrow \infty \), we use a standard argument. □

Let us compute \( \chi(Q^1(a, a + 1)) \) as in the introduction. From [16], we recall the following:

**Theorem 4.** (See [16, Theorem 6.6].)
\[
\chi(K^m(a, a + 1)) = \frac{m}{(a + 1)((m - 1)a + m)} \left( \frac{(m - 1)^2a + (m - 1)m}{a} \right).
\]

By Theorem 1, we have the following:

**Corollary 5.**
\[
\chi(Q^1(a, a + 1)) = \lim_{m \rightarrow \infty} \chi(K^m(a, a + 1)) = (a + 1)! (a + 1)^{2+a}. \]

**Remark 1.** With the formula of \( \chi(K^m(2, 2a + 1)) \) in [10], Manschot has proved
\[
\chi(Q^1(2, 2a + 1)) = \frac{(2a + 1)!}{a!^2}.
\]
This sequence and the one in Corollary 5 coincide with A002457 and A066319 at oeis.org.

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**References**