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Complex Analysis

# Global and local definition of the Monge–Ampère operator on compact Kähler manifolds

Définition globale et locale de l'opérateur de Monge–Ampère sur les variétés kählériennes compactes

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#### ABSTRACT

The aim of this Note is to give a sufficient condition in order for a function in the global domain of definition of the Monge–Ampère operator not to belong to the local domain of the former in the sense of Cegrell, when one looks at the *n*-dimensional complex projective space. Using this result, we show that the subsolution theorem is false for functions in the local domain of definition of the Monge–Ampère operator on such a projective space.

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# RÉSUMÉ

Le but de cet article est de donner une condition suffisante pour qu'une fonction dans le domaine global de définition de l'opérateur Monge–Ampère n'appartienne pas au domaine local de celui-ci dans le sens de Cegrell, lorsqu'on se place sur un espace projectif complexe de dimension *n*. En utilisant ce résultat, nous montrons que le théorème de sous-solution est faux pour des fonctions dans le domaine local de définition de l'opérateur Monge–Ampère sur un tel espace projectif.

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# 1. Introduction

Let *X* be a compact Kähler manifold of complex dimension *n* with a fundamental form  $\omega = \omega_X$ . In [14] V. Guedj and A. Zeriahi have introduced a class  $\mathcal{E}(X, \omega)$  such that for every  $\varphi \in \mathcal{E}(X, \omega)$  one can define the complex Monge–Ampère operator  $(dd^c\varphi + \omega)^n$  globally. They also proved that for all measures  $\mu$  on *X* vanishing on pluripolar sets and  $\mu(X) = 1$ , there exists  $\varphi_{\mu} \in \mathcal{E}(X, \omega)$  with  $\sup_X \varphi_{\mu} = 0$  and  $(dd^c\varphi_{\mu} + \omega)^n = \mu$ . Next, in [12], S. Dinew has shown that the above solution  $\varphi_{\mu}$  is unique. Following ideas and techniques of Cegrell [4] we introduce the class

 $DMA_{loc}(X, \omega) = \{ \varphi \in PSH^{-}(X, \omega) : \forall z \in X, \exists D \ni z, \text{ with } \varphi + \theta \in \mathcal{E}(D) \text{ and } \omega = dd^{c}\theta \text{ on } D \}$ 

where *D* is an open neighbourhood of *z*. By [4], it follows that  $\varphi \in DMA_{loc}(X, \omega)$  if and only if  $\omega_{\varphi}^{n} = (dd^{c}\varphi + \omega)^{n}$  can be defined locally. In this Note we will give a sufficient condition on  $\mu$  such that  $\varphi_{\mu} \notin DMA_{loc}(\mathbb{CP}^{n}, \omega)$ . From this result, we show that for every measure  $\nu$  on the complex projective space  $\mathbb{CP}^{n}$  with  $\nu(\mathbb{CP}^{n}) < 1$  vanishing on pluripolar sets,

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and for every open set  $D \in \mathbb{CP}^n$  we can find  $f \in L^1(\omega^n)$  satisfying supp  $f \subset D$ ,  $\int_{\mathbb{CP}^n} f\omega^n = 1 - \nu(\mathbb{CP}^n)$  and  $\varphi_{\nu+f\omega^n} \notin DMA_{\text{loc}}(\mathbb{CP}^n, \omega)$ . Note that in the case of hyperconvex domains, the subsolution theorem is valid for the class  $DMA_{\text{loc}}$  (see [1]). However, at the end of the Note, we show that the subsolution theorem is false for the class  $DMA_{\text{loc}}(\mathbb{CP}^n, \omega)$ .

#### 2. Preliminaries

In this section we recall the class  $\mathcal{E}(X, \omega)$  introduced and investigated by V.C. Guedj and A. Zeriahi recently (see [14]). Let X be a compact Kähler manifold of complex dimension n and  $\omega$  be a positive closed (1, 1)-current such that  $\int_X \omega^n = 1$ . We refer readers to paper [13] about the notion of  $\omega$ -plurisubharmonic functions. Assume that  $\varphi$  is an  $\omega$ -plurisubharmonic function. We use the notation  $\omega_{\varphi} = dd^c \varphi + \omega$ . By the results of [2] the Monge-Ampère operator  $\omega_{\varphi}^n = (dd^c \varphi + \omega)^n = \omega_{\varphi} \wedge \cdots \wedge \omega_{\varphi}$  is well defined for bounded  $\omega$ -psh functions. From [14] we know that the sequence of measures  $1_{\{\varphi > -j\}}(dd^c \max(\varphi, -j) + \omega)^n$  is increasing and one defines

$$\mathcal{E}(X,\omega) = \left\{ \varphi \in PSH(X,\omega) \colon \lim_{j \to \infty} \int_{X} \mathbb{1}_{\{\varphi > -j\}} \left( dd^{c} \max(\varphi, -j) + \omega \right)^{n} = \int_{X} \omega^{n} \right\}$$

Then one defines  $\omega_{\varphi}^{n} = (dd^{c}\varphi + \omega)^{n} = \lim_{j\to\infty} \mathbb{1}_{\{\varphi>-j\}}(dd^{c}\max(\varphi, -j) + \omega)^{n}$ . Note that Monge–Ampère measures of functions from  $\mathcal{E}(X, \omega)$  do not charge pluripolar sets. We refer to [3,8–11,16–18] for further information about the complex Monge–Ampère equation.

# 3. Auxiliary results

This section is devoted to present some auxiliary results which are needed for the main results in the next section.

**Proposition 3.1.** Let  $\Omega$  be a hyperconvex domain in  $\mathbb{C}^n$  and  $u \in \mathcal{E}(\Omega)$ ,  $v \in PSH^-(\Omega)$ ,  $\alpha \in (0, 1)$  satisfying  $u \ge -|v|^{\alpha}$ . Then  $u \in \mathcal{E}^a(\Omega)$ , where  $\mathcal{E}^a(\Omega)$  denotes the set of  $u \in \mathcal{E}(\Omega)$  such that  $(dd^c u)^n$  vanishes on pluripolar sets of  $\Omega$ .

**Proof.** We may assume that  $u \in \mathcal{F}(\Omega)$ . Assume that *E* is a compact subset in  $\{v = -\infty\}$ . It suffices to prove that  $\int_{F} (dd^{c}u)^{n} = 0$ .

By Lemma 4.3 in [1] there exists  $u_E \in \mathcal{F}(\Omega)$  such that  $u_E \ge u$  and  $(dd^c u_E)^n = 1_E (dd^c u)^n$ . We show that  $u_E = 0$ . Take  $\varepsilon > 0$  and put  $\tilde{u} = \max(u_E, \varepsilon v)$ . Then  $\tilde{u} \ge u_E$  and  $\tilde{u} = u_E$  on the set  $\{v < -(\frac{1}{\varepsilon})^{\frac{1}{1-\alpha}}\}$ . It follows that  $(dd^c \tilde{u})^n = (dd^c u_E)^n = 1_E (dd^c u)^n$ , on the set  $\{v < -(\frac{1}{\varepsilon})^{\frac{1}{1-\alpha}}\}$ . Thus we infer that  $(dd^c \tilde{u})^n \ge (dd^c u_E)^n$  on  $\Omega$ . On the other hand, since  $\int_{\Omega} (dd^c \tilde{u})^n \le \int_{\Omega} (dd^c u_E)^n$  it follows that  $(dd^c \tilde{u})^n = (dd^c u_E)^n$ . Proposition 3.1 in [15] implies that  $u_E = \tilde{u} \ge \varepsilon v$  (see also Theorem 3.15 in [5]). Letting  $\varepsilon \to 0$  we are done. The proof is complete.  $\Box$ 

**Proposition 3.2.** Let  $\Omega$  be a hyperconvex domain in  $\mathbb{C}^n$  and  $u \in \mathcal{E}^a(\Omega)$ . Then for every compact set  $K \subseteq \Omega$  and t > 0 the following holds:

$$\operatorname{cap}(\{u < -t\} \cap K) = \frac{o(1)}{t^n}.$$

**Proof.** By [4], we can choose a function  $\tilde{u} \in \mathcal{F}(\Omega)$  such that  $\tilde{u} \ge u$  on  $\Omega$  and  $\tilde{u} = u$  on K. On the other hand, Lemma 4.1 in [1] implies that  $\tilde{u} \in \mathcal{F}^a(\Omega)$  where  $\mathcal{F}^a(\Omega)$  is the set of functions  $u \in \mathcal{F}(\Omega)$  such that  $(dd^c u)^n$  vanishes on pluripolar sets. By the proof of Proposition 3.4 in [6], we get

$$\operatorname{cap}(\{u < -t\} \cap K) \leq \operatorname{cap}(\{\tilde{u} < -t\}) \leq \frac{2^n \int_{\{\tilde{u} < -\frac{t}{2}\}} (dd^c \tilde{u})^n}{t^n}$$

and the proof follows.  $\hfill \Box$ 

**Corollary 3.3.** Take  $\alpha_1, \ldots, \alpha_k \in (0, 1]$  satisfying  $\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_k} \leq n$  with  $1 \leq k < n$  and put

$$\triangle^n(\mathbf{0},r) = \left\{ z \in \mathbb{C}^n \colon |z_1| < r, \dots, |z_n| < r \right\}, \quad \forall r > \mathbf{0}$$

Then the function  $u(z) = \max(-|\ln |z_1||^{\alpha_1}, \dots, -|\ln |z_k||^{\alpha_k}) \notin \mathcal{E}(\triangle^n(0, r))$  whenever 0 < r < 1.

**Proof.** For  $t \ge 1$  and  $\rho < r$ , we have

$$\operatorname{cap}(\{u < -t\} \cap \triangle^{n}(0, \varrho)) = \frac{\operatorname{cap}(\triangle_{\varrho})^{n-k}}{t^{\frac{1}{\alpha_{1}} + \dots + \frac{1}{\alpha_{k}}}} \ge \frac{\operatorname{cap}(\triangle_{\varrho})^{n-k}}{t^{n}}$$

Assume that  $u \in \mathcal{E}(\triangle^n(0, r))$ . Then  $u \in \mathcal{E}^a(\triangle^n(0, r))$  and Proposition 3.2 implies that  $\operatorname{cap}(\{u < -t\} \cap \triangle^n(0, \varrho)) = \frac{o(1)}{t^n}$  for some  $\varrho < r$ . We get a contradiction and the proof follows.  $\Box$ 

## 4. Main results

Assume that  $\Omega$  is a hyperconvex domain in X. This means that  $\Omega$  is biholomorphic to a hyperconvex domain in  $\mathbb{C}^n$ . Let  $\mu$  be a finite positive Borel measure on X vanishing on pluripolar sets. Then from Lemma 5.14 in [4] there exists a function  $\varphi_{\mu,\Omega} \in \mathcal{F}^a(\Omega)$  such that  $(dd^c \varphi_{\mu,\Omega})^n = \mu$ . In order to obtain the main result of this section, we need the following:

**Proposition 4.1.** Assume that  $\omega = dd^c \theta$  on  $\Omega$  with  $\theta \in PSH^- \cap L^{\infty}(\Omega)$ , where  $\Omega$  is a hyperconvex domain in X. Then for every  $\varphi \in \mathcal{E}(X, \omega)$  the following holds:  $\varphi|_{\Omega} + \theta \leq \varphi_{\mu,\Omega}$ .

**Proof.** Put  $\varphi_i = \max(\varphi, -j)$  and  $\mu = \omega_{\omega}^n$ . Then  $\mathbb{1}_{\{\varphi > -j\}} (dd^c \varphi_i + \omega)^n \nearrow \mu$  on X. Choose  $\psi_i \in \mathcal{F}^a(\Omega)$  such that

$$\left(dd^{c}\psi_{j}
ight)^{n}=1_{\{arphi>-j\}}\left(dd^{c}arphi_{j}+\omega
ight)^{n}
earrow\mu=\left(dd^{c}arphi_{\mu,\varOmega}
ight)^{n}\quad ext{on }\Omega.$$

Using the comparison principle (see Theorem 5.15 in [4]), we have  $\psi_j \searrow \varphi_{\mu,\Omega}$  on  $\Omega$ . We have  $(dd^c(\varphi_j + \theta))^n = (dd^c\varphi_j + \omega)^n \ge (dd^c\psi_j)^n$ . Using the comparison principle, it follows that  $\varphi_j + \theta \le \psi_j$  on  $\Omega$ . Letting  $j \to \infty$  we get  $\varphi|_{\Omega} + \theta \le \varphi_{\mu,\Omega}$  and the desired conclusion follows.  $\Box$ 

We now state and prove our main result:

**Theorem 4.2.** Let  $(\mathbb{CP}^n, \omega)$  be the complex projective space, where  $\omega$  is the Fubini–Study form. Take  $h \notin \mathcal{E}(\triangle^n(0, r))$  (according to Corollary 3.3). Then if

$$\varphi_{\mu,\triangle^n(0,r)} \leqslant A \max(\log |z_1|, h(z_2, \ldots, z_n)) + C,$$

for suitable constants A > 1 and C > 0, we have  $\varphi_{\mu} \notin DMA_{loc}(\mathbb{CP}^n, \omega)$ .

**Proof.** Put  $z' = (z_2, ..., z_n)$ . According to ideas exposed in [7] we show that  $\varphi_{\mu}(z) \leq \frac{1}{4}(A-1)h(z') + C_1$ , on  $\triangle^n(0, r)$ , where  $C_1$  is a constant. Let  $\theta = \frac{1}{2}\log(1+|z|^2) \in PSH \cap C^{\infty}(\mathbb{C}^n)$  such that  $\omega = dd^c\theta$ . For each |z'| < r, put

$$u_{z'}(z_1) = \varphi_{\mu}(z_1, z') + \frac{1}{2}\log(1 + |z_1|^2 + |z'|^2)$$

and  $t_{z'} = e^{h(z')}$ . Since  $u_{z'} \in \mathcal{L}(\mathbb{C})$ , we have  $\int_{\{z_1 \in \mathbb{C}\}} \Delta u_{z'} \leq 1$ . If  $|z_1| \leq t_{z'}$ , then by Proposition 4.1

$$u_{z'}(z_1) \leq \varphi_{\mu, \Delta^n(0,r)} \leq A \max(\log t_{z'}, h(z')) + C = Ah(z') + C \leq \frac{1}{4}(A-1)h(z') + C_1$$

Hence, taking  $C_1 \ge C$  we are done. Assume now that  $t_{z'} < |z_1| < r$ . Since  $\{\zeta \in \mathbb{C}: |\zeta| < r\} \subset \{\zeta \in \mathbb{C}: |\zeta - z_1| < 2r\} \subset \{\zeta \in \mathbb{C}: |\zeta| < 3r\}$ , the Jensen formula implies

$$\begin{split} u_{z'}(z_1) &\leqslant \frac{1}{\pi (2r)^2} \int_{\{|\zeta - z_1| < 2r\}} u_{z'}(\zeta) \, \mathrm{d}V_2(\zeta) \leqslant \frac{1}{4} \frac{1}{\pi r^2} \int_{\{|\zeta| < r\}} u_{z'}(\zeta) \, \mathrm{d}V_2(\zeta) + \frac{1}{2} \log(1 + (n+8)r^2) \\ &\leqslant \frac{1}{4} \bigg[ \frac{1}{2\pi r} \int_{|\zeta| = r} u_{z'}(\zeta) \, \mathrm{d}\sigma(\zeta) - \frac{1}{2\pi t_{z'}} \int_{|\zeta| = t_{z'}} u_{z'}(\zeta) \, \mathrm{d}\sigma(\zeta) + \frac{1}{2\pi t_{z'}} \int_{|\zeta| = t_{z'}} u_{z'}(\zeta) \, \mathrm{d}\sigma(\zeta) \bigg] + \frac{1}{2} \log(1 + (n+8)r^2) \\ &\leqslant \frac{1}{4} \bigg[ \int_{\{|\zeta| < r\}} \log \frac{r}{t_{z'}} \Delta u_{z'} + Ah(z') + C \bigg] + \frac{1}{2} \log(1 + (n+8)r^2) \\ &\leqslant \frac{1}{4} \bigg[ \log \frac{r}{|t_{z'}|} + Ah(z') + C \bigg] + \frac{1}{2} \log(1 + (n+8)r^2) \\ &= \frac{1}{4} (A-1)h(z') + \frac{1}{4} [\log r + C] + \frac{1}{2} \log(1 + (n+8)r^2) \leqslant \frac{1}{4} (A-1)h(z') + C_1. \end{split}$$

Hence,  $\varphi_{\mu}(z) \leq \frac{1}{4}(A-1)h(z') + C_1$  on  $\triangle^n(0,r)$ . However, since  $h \notin \mathcal{E}(\triangle^n(0,r))$  it follows that  $\varphi_{\mu} \notin \mathcal{E}(\triangle^n(0,r))$  and the proof of the theorem is complete.  $\Box$ 

From the above theorem we get the following:

**Corollary 4.3.** Let  $\nu$  be a measure on  $\mathbb{CP}^n$  vanishing on pluripolar sets and satisfying  $\nu(\mathbb{CP}^n) < 1$ . Then for every open subset  $D \Subset \mathbb{CP}^n$  there exists a function  $f \in L^1(\omega^n)$  satisfying supp  $f \subset D$ ,  $\int_{\mathbb{CP}^n} f \omega^n = 1 - \nu(\mathbb{CP}^n)$  and  $\varphi_{\nu+f\omega^n} \notin DMA_{loc}(\mathbb{CP}^n, \omega)$ .

**Proof.** Without loss of generality we may assume that  $D = \triangle^n(0, r_0)$  with  $0 < r_0 < 1$ . Put

$$\Phi(z) = \ln(|z_1|^2 + \dots + |z_{n-1}|^2 + e^{-|\ln|z_n||^{\frac{1}{2}}}).$$

It follows that  $\Phi \in PSH \cap C^{\infty}(\triangle^n(0, 1) \setminus \bigcup_{j=1}^n \{z_j = 0\})$ . From the obvious bound

$$\Phi(z) = \max\left(\ln|z_1|, \dots, \ln|z_{n-1}|, -\left|\ln|z_n|\right|^{\frac{1}{2}}\right) + O(1)$$

and from Proposition 3.1 we infer that  $\Phi \in \mathcal{E}^{a}(\Delta^{n}(0, 1))$ . On the other hand, as  $(dd^{c}\Phi)^{n}|_{\bigcup_{i=1}^{n}\{z_{i}=0\}}=0$ , it follows that

$$\left(dd^{c}\Phi\right)^{n}=1_{\{\Delta^{n}(0,1)\setminus\bigcup_{j=1}^{n}\{z_{j}=0\}\}}\left(dd^{c}\Phi\right)^{n}=g\omega^{n}$$

Since  $\int_{\Delta^n(0,r)} (dd^c \Phi)^n \to 0$  as  $r \to 0$ , we can choose A > 1 and  $r_1 < r_0$  such that  $A^n \int_{\Delta^n(0,r_1)} (dd^c \Phi)^n = 1 - \nu(\mathbb{CP}^n)$ . Put  $\mu = \nu + A^n 1_{\Delta^n(0,r_1)} (dd^c \Phi)^n$ , where  $1_E$  is the characteristic function of E. We will show that  $\varphi_\mu \notin DMA_{\text{loc}}(\mathbb{CP}^n, \omega)$ . Indeed, we have  $(dd^c \varphi_{\mu,\Delta^n(0,r_1)})^n \ge (dd^c (A\Phi))^n$ ,  $n \to \Delta^n(0,r_1)$ . The comparison principle implies that  $\varphi_{\mu,\Delta^n(0,r_1)} \le A\Phi - \inf_{\partial \Delta^n(0,r_1)} A\Phi = A\Phi + C$ . On the other hand, by Corollary 3.3, we have

$$h(z') = \max(\ln |z_2|, ..., \ln |z_{n-1}|, -|\ln |z_n||^{\frac{1}{2}}) \notin \mathcal{E}(\triangle^n(0, r_1)).$$

Now applying Theorem 4.2 we get  $\varphi_{\mu} \notin DMA_{loc}(\mathbb{CP}^n, \omega)$  and the desired conclusion follows.  $\Box$ 

**Remark 4.4.** Let  $\omega$  be a Fubini–Study form on  $\mathbb{CP}^n$ . Now we construct a measure  $\mu$  on  $\mathbb{CP}^n$  and a function  $\psi \in PSH(\mathbb{CP}^n, \omega) \cap L^{\infty}_{loc}(\mathbb{CP}^n \setminus \{0\})$  (by the results of J.-P. Demailly in [10] we know that  $\psi \in DMA_{loc}(\mathbb{CP}^n, \omega)$ ) such that  $\mu \leq C(dd^c\psi + \omega)^n$  for some constant C > 1 and  $(dd^c\psi + \omega)^n$  vanishes on pluripolar sets but  $\varphi_{\mu} \notin DMA_{loc}(\mathbb{CP}^n, \omega)$ . This shows that the subsolution theorem is false for the class  $DMA_{loc}(\mathbb{CP}^n, \omega)$ .

Consider  $h_n \in \mathcal{L}(\mathbb{C})$  by  $h_n(z_n) = \ln |z_n|$  if  $|z_n| \ge \frac{1}{e}$  and  $h_n(z_n) = -|\ln |z_n||^{\frac{1}{2}}$  if  $|z_n| \le \frac{1}{e}$ . Set  $\psi(z) = \max(\ln |z_1|, \dots, \ln |z_{n-1}|, h_n(z_n)) - \frac{1}{2}\ln(|z|^2 + 1)$  if  $z \in \mathbb{C}^n$  and  $\psi(z) = \limsup_{w \in \mathbb{C}^n, w \to z} \psi(w)$  if  $z \in \mathbb{CP}^n \setminus \mathbb{C}^n$ . We have  $\psi \in PSH(\mathbb{CP}^n, \omega) \cap L^{\infty}_{loc}(\mathbb{CP}^n \setminus \{0\})$ . By Proposition 3.1, we infer that  $\psi \in \mathcal{E}^a(\mathbb{C}^n)$ . As in Corollary 4.3 we choose A > 1 and r > 0 such that  $A^n \int_{\Delta^n(0,r)} (dd^c \psi + \omega)^n = 1$ . Set  $\mu = A^n \mathbb{1}_{\Delta^n(0,r)} (dd^c \psi + \omega)^n$ . From Theorem 4.2 we get  $\varphi_\mu \notin DMA_{loc}(\mathbb{CP}^n, \omega)$ .

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