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Lie Algebras

The *t*-analog of the level one string function for twisted affine Kac–Moody algebras

Le t-analogue de la fonction corde de niveau un pour les algèbres de Kac–Moody affines tordues

Sachin S. Sharma, Sankaran Viswanath

The Institute of Mathematical Sciences, CIT campus, Taramani, Chennai 600113, India

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ABSTRACT

We study Lusztig's *t*-analog of weight multiplicities, or affine Kostka–Foulkes polynomials, associated to level one representations of twisted affine Kac–Moody algebras. We obtain an explicit closed form expression for the unique *t*-string function, using constant term identities of Macdonald and Cherednik. This extends previous work on *t*-string functions for the untwisted simply-laced affine Kac–Moody algebras.

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RÉSUMÉ

On étudie le *t*-analogue, d'après Lusztig, des multiplicités des poids, c'est-à-dire les polynômes de Kostka–Foulkes affines, associés aux représentations du niveau un des algèbres de Kac–Moody affines tordues. On obtient une expression explicite pour l'unique *t*-fonction de corde, en utilisant les identités de Macdonald et Cherednik. Cela étend des travaux précédents sur les *t*-fonctions de corde pour les algèbres de Kac–Moody affines non-tordues de type A-D-E.

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1. Introduction

Let g be a twisted affine Kac–Moody algebra of rank l + 1 ($l \ge 1$). Let its root space decomposition be given by

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} (\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha})$$

where Δ_+ is the set of positive roots, and let $\operatorname{mult} \alpha := \operatorname{dim}(\mathfrak{g}_{\alpha})$ be the root multiplicity of α . We let $\overline{\mathfrak{g}}$ denote the underlying finite dimensional simple Lie algebra of rank *l* [4].

For a dominant integral weight λ , let $L(\lambda)$ denote the irreducible g-module of highest weight λ . The $L(\lambda)$ are integrable g-modules in category \mathcal{O} . Their formal characters are given by the Kac–Weyl character formula.

Now, let *t* be an indeterminate. In this Note, we study Lusztig's *t*-analog of weight multiplicity (or *affine Kostka–Foulkes polynomial*) $K_{\lambda\mu}(t)$. Given a dominant weight λ of positive level, and a dominant weight μ of $L(\lambda)$, $K_{\lambda\mu}(t)$ is defined to be the following alternating sum over the Weyl group *W* of \mathfrak{g} :

E-mail addresses: sachin@imsc.res.in (S.S. Sharma), svis@imsc.res.in (S. Viswanath).

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$$K_{\lambda\mu}(t) := \sum_{w \in W} \epsilon(w) \mathcal{P} \big(w(\lambda + \rho) - (\mu + \rho); t \big)$$

where $\epsilon(w)$ is the sign of w, and $\mathcal{P}(\beta; t)$ is the *t*-analog of the Kostant partition function of \mathfrak{g} , defined to be the coefficient of e^{β} in the product $\prod_{\alpha \in \Delta_+} (1 - te^{\alpha})^{-\text{mult}\alpha}$.

The $K_{\lambda\mu}(t)$ have several important properties [2,7,8]: (a) they are the transition coefficients between the bases of affine Hall–Littlewood functions and the formal characters of the $L(\lambda)$, (b) they are polynomials with non-negative integral coefficients, and (c) $K_{\lambda\mu}(1) = \dim L(\lambda)_{\mu}$.

Let δ be the null root of \mathfrak{g} , and let μ now denote a *maximal* dominant weight of $L(\lambda)$. To understand the structure of the module $L(\lambda)$, one studies the generating functions of its weight multiplicities along δ -strings: $a_{\mu}^{\lambda}(q) := \sum_{k \ge 0} \dim(L(\lambda)_{\mu-k\delta})q^k$. These are (up to a factor of a power of q) the *string functions* of the module $L(\lambda)$, and are known to be modular forms for certain congruence subgroups of $SL_2\mathbb{Z}$ [4]. Now for the *t*-analog, in view of property (c) above, it is natural to consider the generating function

$$a_{\mu}^{\lambda}(t,q) := \sum_{k \geqslant 0} K_{\lambda,\mu-k\delta}(t)q^k$$

of the $K_{\lambda\mu}(t)$ along δ -strings. Following [8], the $a^{\lambda}_{\mu}(t,q)$ will be referred to as *t*-string functions.

Among the nontrivial irreducible integrable modules in category \mathcal{O} , the *basic representation* $L(\Lambda_0)$ is the most important. For the affine Lie algebras which are (untwisted) simply-laced, or twisted, all level one irreducible integrable representations in category \mathcal{O} are obtained from the basic representation by the action of an automorphism of the affine Dynkin diagram, followed by tensoring with a one-dimensional representation. The following classical result describes the string functions of the basic representation [5]:

Theorem 1.1. If g is an untwisted simply-laced affine Lie algebra, or a twisted affine algebra, the basic representation admits a unique string function $a_{A_0}^{A_0}(q)$, given by

$$a_{\Lambda_0}^{\Lambda_0}(q) = \prod_{n=1}^{\infty} (1-q^n)^{-\operatorname{mult}(n\delta)}.$$

The remaining affine algebras are the untwisted ones of types B, C, F, G. In these cases, there are multiple string functions associated with the basic representation, and their description is somewhat more involved [5].

Now, the *t*-analogs of string functions were studied in [8], where they were related to the constant term identities arising in the theory of Macdonald polynomials, via the following observation:

Lemma 1.2. $a_{\mu}^{\lambda}(t,q) = ct(\tilde{\Delta}e^{-\mu}\chi_{\lambda})$ where ct is the constant term functional, $\tilde{\Delta}$ is the modified Cherednik kernel and χ_{λ} is the formal character of the representation $L(\lambda)$.

When g is untwisted simply-laced, the *t*-string function $a_{A_0}^{A_0}(t,q)$ of the basic representation can be computed in closed form using the above lemma and Cherednik's Macdonald–Mehta constant term identity [8]:

Theorem 1.3. Let \mathfrak{g} be an untwisted simply-laced affine algebra, i.e., $\mathfrak{g} = A_l^{(1)}$ $(l \ge 1)$, $D_l^{(1)}$ $(l \ge 4)$ or $E_l^{(1)}$ (l = 6, 7, 8). Then, we have

$$a_{\Lambda_0}^{\Lambda_0}(t,q) = \prod_{n=1}^{\infty} \prod_{i=1}^{l} \left(1 - t^{e_i + 1}q^n\right)^{-1}$$

where e_1, \ldots, e_l are the exponents of $\bar{\mathfrak{g}}$.

The main theorem of this article is the corresponding result for the twisted affine algebras.

2. The main theorem

For later use in stating our main theorem, we first recall relevant facts concerning generalized exponents. Let $V = V(\lambda)$ be the irreducible finite dimensional representation with highest weight λ of a finite dimensional simple Lie algebra \bar{g} . We fix a triangular decomposition $\bar{g} = \bar{n}^- \oplus \bar{b} \oplus \bar{n}^+$, and choose a regular (principal) nilpotent element $E \in \bar{n}^+$. Let V_0 denote the zero weight space of V, and define the Brylinski-Kostant filtration [1,6] of V_0 via $\mathcal{F}^{(p)}(V_0) := \ker(E^p) \cap V_0$ for $p \ge 0$. Then the multiset $\mathbb{E}(V)$ of generalized exponents of V contains each $p \ge 0$ as many times as $\dim(\mathcal{F}^{(p+1)}(V_0)/\mathcal{F}^{(p)}(V_0))$. In other words, the Hilbert series of the Brylinski-Kostant filtration of V_0 is the generating series of generalized exponents of V:

$$\sum_{v\geq 0} \dim \left(\mathcal{F}^{(p+1)}(V_0) / \mathcal{F}^{(p)}(V_0) \right) t^p = \sum_{k\in\mathbb{E}(V)} t^k.$$

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This generating series is also equal to the Kostka–Foulkes polynomial $K_{\lambda,0}(t)$ of the Lie algebra \bar{g} .

The following classical result (see, for e.g., [3]) provides a purely combinatorial characterization of generalized exponents in important special cases, and is one of the ingredients in the proof of our main theorem:

Proposition 2.1. Let \bar{g} be a finite dimensional simple Lie algebra, and let θ_l and θ_s denote its highest long and short roots respectively (taking these to be equal if there is only one root length). For each $k \ge 1$, let n(k) (resp. $n_s(k)$) denote the number of roots (resp. short roots) of height k. Then, for each $j \ge 1$, the number of times j occurs as a generalized exponent of $V(\theta_l)$ (resp. $V(\theta_s)$) equals n(j) - n(j + 1) (resp. $n_s(j) - n_s(j + 1)$).

Now, let g be a twisted affine algebra of type $X_N^{(r)}$; here X_N is a simply laced (A-D-E) Dynkin diagram of finite type, with a diagram automorphism σ of order r (r = 2 or 3). Let m denote the finite dimensional simple Lie algebra with Dynkin diagram X_N , and let σ also denote the automorphism of m induced by the diagram automorphism. For each $k \in \mathbb{Z}$, let \mathfrak{m}_k be the eigenspace of σ for the eigenvalue $\exp(2\pi k i/r)$. Since σ acts diagonalizably on \mathfrak{m} , we have a $\mathbb{Z}/r\mathbb{Z}$ gradation: $\mathfrak{m} = \bigoplus_{i \in \mathbb{Z}/r\mathbb{Z}} \mathfrak{m}_i$. Further, if \mathfrak{h} is a Cartan subalgebra of \mathfrak{m} , let $\mathfrak{h}_i := \mathfrak{h} \cap \mathfrak{m}_i$ for all $j \in \mathbb{Z}$.

The Lie algebra \mathfrak{g} has a realization as the universal central extension of an equivariant loop algebra [4, Chapter 8]: $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} (z^k \otimes \mathfrak{m}_k) \oplus \mathbb{C}K \oplus \mathbb{C}d$ where K is the canonical central element, d is the degree derivation and z is an indeterminate (the loop coordinate). There is a natural \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ with $\mathfrak{g}_0 = \mathfrak{m}_0 + \mathbb{C}K + \mathbb{C}d$ and $\mathfrak{g}_j = z^j \otimes \mathfrak{m}_j$ for $j \neq 0$. This has the following properties [4]: (a) \mathfrak{g}_0 is reductive, with semisimple part \mathfrak{m}_0 , (b) for all $j \neq 0$, \mathfrak{g}_j is an irreducible \mathfrak{m}_0 -module (say, with highest weight λ_j), (c) if $\mathfrak{g} \neq A_{2l}^{(2)}$, then $\lambda_j = \theta_l$ (resp. θ_s) when r|j (resp. $r \nmid j$), where θ_l (resp. θ_s) is the highest long (resp. short) root of \mathfrak{m}_0 , and (d) if $\mathfrak{g} = A_{2l}^{(2)}$, then $\lambda_j = \theta_l$ (resp. $2\theta_s$) when r|j (resp. $r \nmid j$).

Let \mathbb{E}_n denote the multiset of generalized exponents of the \mathfrak{m}_0 -module \mathfrak{g}_n for n > 0. The main result of this Note is the following:

Theorem 2.2. Let \mathfrak{g} be a twisted affine algebra. The t-string function of the basic representation of \mathfrak{g} is given by

$$a_{\Lambda_0}^{\Lambda_0}(t,q) = \prod_{n=1}^{\infty} \prod_{e \in \mathbb{E}_n} (1 - t^{e+1}q^n)^{-1}.$$

We observe from the definitions that the cardinality of \mathbb{E}_n is the dimension of the zero weight space of \mathfrak{g}_n as an \mathfrak{m}_0 -module, i.e., $|\mathbb{E}_n| = \dim(z^n \otimes \mathfrak{h}_n)$. Since $z^n \otimes \mathfrak{h}_n$ is the root space of \mathfrak{g} corresponding to the imaginary root $n\delta$, we obtain $|\mathbb{E}_n| = \operatorname{mult}(n\delta)$. Thus at t = 1, Theorem 2.2 reduces to the classical result of Theorem 1.1.

We also remark that when g is simply laced, the g_j (j > 0) are all isomorphic to the adjoint representation of \mathfrak{m}_0 . In this case, the generalized exponents coincide with the usual exponents of \mathfrak{m}_0 . Theorem 2.2 is thus a generalization of Theorem 1.3.

Next, we derive an interesting corollary. If \mathfrak{g} is an affine Kac–Moody algebra of rank l + 1, and e_i , f_i (i = 0, ..., l) are the Chevalley generators, the principal Heisenberg subalgebra \mathfrak{s} of \mathfrak{g} is defined to be $\mathfrak{s} := \{x \in \mathfrak{g}: [x, \sum_{i=0}^{l} e_i] \in \mathbb{C}K\}$ where K is the central element of \mathfrak{g} [4]. The principal gradation of \mathfrak{g} induces a gradation $\mathfrak{s} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{s}_j$. The *exponents of the affine algebra* \mathfrak{g} are the elements of the (infinite) multiset of nonzero integers in which each j occurs dim \mathfrak{s}_j times. We now have the following corollary to Theorem 2.2, which relates a certain specialization of the basic *t*-string function to the exponents of the affine algebra and its underlying finite dimensional simple Lie algebra:

Corollary 2.3. Let g be a twisted affine algebra or an untwisted simply-laced affine algebra, with Coxeter number h. Let \tilde{g} be its underlying finite dimensional simple Lie algebra. Then

$$a_{\Lambda_0}^{\Lambda_0}(q,q^h) = \frac{\prod_{\bar{e}\in\mathbb{E}(\bar{\mathfrak{g}})}(1-q^{\bar{e}+1})}{\prod_{e\in\mathbb{E}(\mathfrak{g}), e>0}(1-q^{e+1})}$$
(1)

where $\mathbb{E}(\bar{\mathfrak{g}})$ and $\mathbb{E}(\mathfrak{g})$ are the multisets of exponents of $\bar{\mathfrak{g}}$ and \mathfrak{g} respectively.

The details of the proofs of Theorem 2.2 and the corollary will appear elsewhere. We content ourselves here with some brief remarks about the proof. As in [8], the starting point is Lemma 1.2. For g not of type $A_{2l}^{(2)}$, we use Cherednik's computation of the Macdonald–Mehta type constant term, together with the combinatorial characterization of generalized exponents given by Proposition 2.1. For the $g = A_{2l}^{(2)}$ case, we first derive a Macdonald–Mehta type identity by suitably specializing the Macdonald constant term for the non-reduced affine root system of type (C_n^{\vee}, C_n) , and use this to complete the proof.

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