1. Introduction

The collisional gain operator for Maxwellian molecules in $\mathbb{R}^N$ is defined by

$$ Q_+ [f, g](v) = \int_{\mathbb{R}^N \times S^{N-1}} B(n \cdot \hat{q}) f(v') g(v'_*) dv_* d\sigma(n). $$

(1)

In the homogeneous Boltzmann equation $\partial_t f = Q_+ [f, f] - f$, the difference $Q_+ [f, f] - f$ accounts for the changes in the velocity distribution $f$ due to binary particle collisions. In (1), the cross section $B : [-1, 1] \rightarrow \mathbb{R}_{\geq 0}$ determines the frequency at which collisions between particles of velocities $v'$ and $v'_*$ occur, and these pre-collisional velocities are related to the post-collisional ones, $v$ and $v_*$, by

$$ v' = \frac{1}{2} (v + v_* + |q|n), \quad v'_* = \frac{1}{2} (v + v_* - |q|n), \quad \text{with} \quad q = v - v_* \quad \text{and} \quad \hat{q} = q/|q|. $$

In Maxwellian gases, $B$ depends on $n \cdot \hat{q}$ but not on $|v - v_*|$, and

$$ \int_{S^{N-1}} B(e \cdot n) d\sigma(n) = 1 \quad \text{for every unit vector} \quad e \in \mathbb{R}^N. $$

(2)
Under the additional assumption that $B$ is even, $B(s) = B(-s)$, it has been shown by Villani [2] that $Q_+$ does not increase the Fisher information, defined on probability densities $f$ by

$$\mathcal{F}[f] = \int_{\mathbb{R}^N} \frac{|\nabla f(x)|^2}{f(x)} \, dx = 4 \int_{\mathbb{R}^N} \left|\nabla \sqrt{f(x)}\right|^2 \, dx.$$  

More precisely, it has been shown that $\mathcal{F}[Q_+[f, g]] \leq (\mathcal{F}[f] + \mathcal{F}[g])/2$. Below, we prove that the hypothesis $B(s) = B(-s)$ can be removed at the price of replacing the original estimate by

$$\mathcal{F}[Q_+[f, g]] \leq \frac{1 + \lambda_B}{2} \mathcal{F}[f] + \frac{1 - \lambda_B}{2} \mathcal{F}[g] \quad \text{with} \quad \lambda_B := \int_{\mathcal{S}^{N-1}} (\mathbf{e} \cdot n) B(\mathbf{e} \cdot n) \, d\sigma(n) \in (-1, 1). \quad (3)$$

For even $B$, we have $\lambda_B = 0$ and thus recover the estimate [2]. Our main contribution in this note, however, is a novel, concise derivation of the following representation formula,

$$\nabla Q_+[f, g](v) = \frac{1}{2} \int_{\mathcal{S}^{N-1} \times \mathbb{R}^N} B(\hat{\mathbf{q}} \cdot n) \left\{ Y_+ + P_{n,\hat{a}}(Y_-) \right\} \, d\sigma(n) \, dv. \quad \text{with} \quad Y_+ := \nabla f(\mathbf{v}') g(\mathbf{v}_a') \pm f(\mathbf{v}') \nabla g(\mathbf{v}_a'). \quad (4)$$

The “projection” operator $P$ is defined for $a, b, x \in \mathbb{R}^N$ by

$$P_{a,b}[x] = (\hat{a} \cdot \hat{b}) x - (\hat{a} \times \hat{b}) \cdot x, \quad \text{with} \quad v \wedge w = vw^T - wv^T, \quad (5)$$

and $\hat{a} = a/|a|$ denotes the normalization of a vector $a \in \mathbb{R}^N$. Notice that $v \wedge w$ is an anti-symmetric $N \times N$-matrix, and in particular, in dimension $N = 3$, one has $(\hat{a} \wedge \hat{b}) \cdot x = (\hat{a} \times \hat{b}) \times x$. Formula (4) — for symmetric $B$ — is at the heart of Villani’s original proof, where it is obtained from integration by parts and geometric considerations. Below, we prove (4) in one line by Fourier methods.

**Notation.** Inside integrals, $f, f', g_+, g_-$ abbreviate $f(v), f'(v), g(v_+), g(v_-)$, and $Q_+$ abbreviates $Q_+[f, g]$.

### 2. Fourier representation

Our starting point is the following famous identity by Bobylëv [1]:

**Lemma 2.1.** Given two probability densities $f$ and $g$, then

$$\hat{Q}_+[f, g](\xi) = \hat{Q}_+[f, g](\xi) = \int_{\mathcal{S}^{N-1}} B(n \cdot \hat{\mathbf{e}}) \hat{f}(\xi_+) \hat{g}(\xi_-) \, d\sigma(n) \quad \text{with} \quad \xi_\pm = \frac{1}{2}(\xi \pm |\xi| n). \quad (6)$$

The key observation is that representation (6) in combination with the elementary relation (7) below — which admits a one-line proof — yields the Fourier analogue of (4).

**Lemma 2.2.** For arbitrary $\xi \in \mathbb{R}^N$, and with $\xi_\pm$ defined in (6),

$$(1 + P_{n,\xi})[\xi_+] + (1 - P_{n,\xi})[\xi_-] = 2\xi. \quad (7)$$

**Proof.** On one hand, $\xi_+ + \xi_- = \xi$ follows directly from (6). And on the other hand, also

$$P_{n,\xi}[\xi_+ - \xi_-] = P_{n,\xi}[|\xi| n] = |\xi| \left( (n \cdot \hat{\xi}) n - n(n \cdot \hat{\xi}) + \hat{\xi}(n \cdot n) \right) = |\xi| \hat{\xi} = \xi,$$

since $n \cdot n = 1$, and $\hat{\xi} := \xi/|\xi|$ by definition. \qed

Inserting the relation (7) under the integral in (6) gives

$$\nabla \hat{Q}_+(\xi) = i\xi \hat{Q}_+(\xi) = \frac{1}{2} \int_{\mathcal{S}^{N-1}} B(\hat{\xi} \cdot n) [(1 + P_{n,\xi})[i\xi_+] + (1 - P_{n,\xi})[i\xi_-]] \hat{f}(\xi_+) \hat{g}(\xi_-) \, d\sigma(n). \quad (8)$$

We shall now show that the Fourier transform of (8) is (4). Substituting

$$\hat{f}(\xi_+) = \int_{\mathbb{R}^N} e^{-iv \cdot \xi_+} f(v) \, dv, \quad i\xi_+ \hat{f}(\xi_+) = \int_{\mathbb{R}^N} e^{-iv \cdot \xi_+} \nabla f(v) \, dv,$$
and respective expressions for \( \hat{\xi}(\xi) \), \( i \xi \cdot \hat{\xi}(\xi) \) under the integral in (8), gives, with \( Y_\pm := \nabla f g_\pm \pm \nabla g \),

\[
\nabla \hat{Q}_+(\xi) = i \xi \cdot \hat{Q}(\xi) = \frac{1}{2} \iiint_{\mathbb{S}^{N-1} \times \mathbb{R}^N \times \mathbb{R}^N} B(\xi \cdot n) \{ Y_+ + P_n,\xi [Y_-] \} e^{-i(\xi \cdot n + v_\perp \cdot \xi)} \, d\sigma(n) \, dv \, dv_\perp
\]

Next, we apply a particular change of variables — which has been designed by Bobylëv [1] — inside the \( n \)-integral to exchange the roles of \( \xi \) and \( q \). For the corresponding treatment of the projection operator, we need

**Lemma 2.3.** For arbitrary vectors \( q, \xi \in \mathbb{R}^N \setminus \{0\} \), and for any measurable function \( A : [-1, 1] \times [-1, 1] \to \mathbb{R} \), one has

\[
\int_{\mathbb{S}^{N-1}} A(\xi \cdot n, \hat{\xi} \cdot n) \hat{\xi} \wedge n \, d\sigma(n) = - \int_{\mathbb{S}^{N-1}} A(\xi \cdot n, \hat{\xi} \cdot n) \hat{\xi} \wedge n \, d\sigma(n).
\]

In fact, both (matrix-valued) integrals are multiples of \( \xi \wedge q \), and vanish if \( \xi \) and \( q \) are linearly dependent.

Before proving (9), we show that it indeed concludes the calculation started above. First, observe that (notice the change of order in the subscripts)

\[
P_{n,\xi} = (n \cdot \hat{\xi}) 1 - n \wedge \hat{\xi} \quad \text{and} \quad P_{q,n} = (n \cdot \hat{q}) 1 + n \wedge \hat{q}.
\]

We substitute (9) under the \( n \)-integral above and observe that its value does not change upon replacing \( n \) by its mirror image in the hyperplane orthogonal to \( \hat{\xi} - \hat{q} \).

\[
\nabla \hat{Q}_+(\xi) = \frac{1}{2} \iiint_{\mathbb{R}^N \times \mathbb{R}^N} \left( \iiint_{\mathbb{S}^{N-1}} e^{i(q \cdot n) B(\xi \cdot n) \{ Y_+ + P_q,\xi [Y_-] \} \, d\sigma(n)} e^{-i(\xi \cdot n + v_\perp \cdot \xi)/2} \, dv \, dv_\perp \right)
\]

Formula (4) is now obtained by performing a change of variables \( (v, v_\perp) \leftrightarrow (v', v'_\perp) \) under the integral. This substitution is of determinant one, it changes \( Y_\pm \) into \( Y'_\pm \), and it exchanges \( \hat{q} \) with \( n \) as desired.

**Proof of Lemma 2.3.** Let \( X \subset \mathbb{R}^N \) be the subspace spanned by \( \xi \) and \( q \). Denote its orthogonal complement by \( X' \). We start by proving the second claim, namely that

\[
I := \int_{\mathbb{S}^{N-1}} A(\xi \cdot n, \hat{\xi} \cdot n) \hat{\xi} \wedge n \, d\sigma(n) \quad \text{and} \quad J := \int_{\mathbb{S}^{N-1}} A(\xi \cdot n, \hat{\xi} \cdot n) \hat{\xi} \wedge n \, d\sigma(n)
\]

are both scalar multiples of \( \xi \wedge q \). Obviously, \( I \) and \( J \) inherit the anti-symmetry of their integrands, so

\[
v^T I w = - w^T I v \quad \text{and} \quad v^T J w = - w^T J v
\]

holds for arbitrary \( v, w \in \mathbb{R}^N \). We shall now show that these products are actually zero whenever \( w \in X' \). Indeed, for \( w \in X' \),

\[
I w = \int_{\mathbb{S}^{N-1}} A(\xi \cdot n, \hat{\xi} \cdot n) \hat{\xi} n^T w \, d\sigma(n).
\]

Perform a change of variables \( n = R^T \hat{n} \) with an orthogonal matrix \( R \) under the integral such that \( R x = x \) for \( x \in X \) and \( R y = -y \) for \( y \in X' \). This change leaves the spherical measure invariant, and the integrand in (11) changes to

\[
A(\hat{\xi} \cdot R^T \hat{n}, \hat{\xi} \cdot R^T \hat{n}) \hat{\xi} \hat{n}^T (R w) = - A(\hat{\xi} \cdot \hat{n}, \hat{\xi} \cdot \hat{n}) \hat{n}^T \hat{\xi} w,
\]

which shows \( I w = - I w = 0 \). Thus \( I \) is an anti-symmetric matrix that is trivial on \( X' \). But the space of anti-symmetric matrices on \( X \) is (at most) one-dimensional, and is spanned by \( q \wedge \xi \). So \( I \) and \( J \) by a similar argument — \( J \) are scalar multiples of \( q \wedge \xi \). To prove (9), consider another orthogonal change of variables \( n = \tilde{R} \hat{n} \), in which \( \tilde{R} \hat{\xi} = \hat{q} \) and \( \tilde{R} \hat{q} = \hat{\xi} \). We find
Lemma 3.1.\]

\[ I = \int_{\mathbb{R}^{N-1}} A(\hat{\eta} \cdot R^T \hat{n}, \hat{\xi} \cdot R^T \hat{n}) \hat{q} \wedge (R^T \hat{n}) \, d\sigma(\hat{n}) = \int_{\mathbb{R}^{N-1}} A(R\hat{q} \cdot \hat{n}, R\hat{\xi} \cdot \hat{n}) R^T ((R\hat{q}) \wedge \hat{n}) \, d\sigma(\hat{n}) = R^T J R. \]

Since \( R^T (q \wedge \xi) R = \xi \wedge q = -q \wedge \xi \), it follows that \( I = -J \). \[ \square \]

3. Estimate on the Fisher information

In order to arrive at (3), we employ (4) in the same way as done in [2]. We adopt the abbreviations \( f' = f(v') \), \( g'_* = g(v'_*) \), etc. By definition of \( Q_+ \), and since \( B \) is a Maxwellian kernel, the quotient \( B(n \cdot \hat{q}) f' g'_*/Q_+[f, g] \) defines \( \varphi \) for every \( v \) — a probability density for integration w.r.t. \( d\nu_* d\sigma(n) \). Now rewrite the quotient \( \nabla Q_+[f, g]/Q_+[f, g] \) using (4) and apply Jensen’s inequality to find

\[ \frac{|\nabla Q_+(v)|^2}{Q_+(v)} \leq \frac{1}{4} \int_{\mathbb{R}^N} \left| \frac{Y'_+ + P_{n,q}[Y'_-]}{f' g'_*} \right|^2 \frac{B(n \cdot \hat{q}) f' g'_*}{Q_+(v)} \, d\nu_* \, d\sigma(n). \]

Multiply this expression by \( Q_+[f, g] \), integrate w.r.t. \( v \), and change variables \( (v, v_*) \leftrightarrow (v', v'_*) \) again to obtain an estimate on the Fisher information:

\[ \mathcal{F}[Q_+[f, g]] \leq \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} B(\hat{q} \cdot n) \frac{|Y_+ + P_{q,n}[Y_-]|^2}{2 f g_*} \, d\tau \, d\nu_* \quad (12) \]

To finish the proof, two properties of the operators \( P \) are needed: the first is simply

\[ P_{q,n} + P^T_{q,n} = 2(\hat{q} \cdot n) 1, \quad (13) \]

which follows from the anti-symmetry \( (\hat{q} \wedge n)^T = -\hat{q} \wedge n \). The second is taken from [2, Lemmata 3 and 4]:

Lemma 3.1. For arbitrary vectors \( a, b \in \mathbb{R}^N \setminus \{0\} \) and \( x \in \mathbb{R}^N \),

\[ |P_{ab}[x]| \leq |x|. \quad (14) \]

Expand the square under the integral in (12), using (14) and (13):

\[ \frac{|Y_+ + P_{q,n}[Y_-]|^2}{2 f g_*} = \frac{|Y_+|^2 + |P_{q,n}[Y_-]|^2 + Y_+ \cdot (P_{q,n} + P^T_{q,n}) [Y_-]|}{2 f g_*} \leq \frac{|Y_+|^2 + |Y_-|^2 + 2(\hat{q} \cdot n) Y_+ \cdot Y_-}{2 f g_*} \]

\[ = \frac{|\nabla \sqrt{f}|^2 g_* + f |\nabla \sqrt{g_*}|^2 + (\hat{q} \cdot n) \left( |\nabla \sqrt{f}|^2 g_* - f |\nabla \sqrt{g_*}|^2 \right)}{2 f g_*}. \]

To arrive at (3), insert this expansion into (12), use the Maxwell property (2), the definition of \( \lambda_B \) in (3), and the fact that \( f \) and \( g \) are probability densities.

References