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Analytic Geometry

Semistability of invariant bundles over G/Γ

Semi-stabilité de fibrés invariants sur G/Γ

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ABSTRACT

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$, and let Γ be a cocompact lattice in G. We prove that any invariant bundle on G/Γ is semistable.

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RÉSUMÉ

Soit Γ un sous-groupe discret cocompact d'un groupe algébique réductif affine G. Nous démontrons que tout fibré invariant sur G/Γ est semi-stable.

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1. Introduction

Let G be a connected complex reductive affine algebraic group, and let $K \subset G$ be a maximal compact subgroup. Fixing a K-invariant Hermitian form on $\mathrm{Lie}(G)$, we may extend it to a right-translation invariant Hermitian structure on G. If ω_G is the corresponding $(1,1)$-form on G, and $\dim_{\mathbb{C}} G = \delta$, we prove that the form $\omega_G^{\delta-1}$ is closed (see Proposition 2.1).

Let $\Gamma \subset G$ be a cocompact lattice. The descent of ω_G to the compact quotient G/Γ will be denoted by $\widetilde{\omega}$. So, $\mathrm{d}\widetilde{\omega}^{\delta-1} = 0$ by Proposition 2.1. This allows us to define the degree of a coherent analytic sheaf on G/Γ; as a consequence, semistable vector bundles on G/Γ can be defined.

A vector bundle E on G/Γ is called invariant if the pullback of E using the left-translation by any element of G is holomorphically isomorphic to E. We prove that invariant vector bundles are semistable (see Lemma 2.4). It may be mentioned that Lemma 2.4 remains valid for holomorphic principal bundles on G/Γ with a reductive group as the structure group.

2. Hermitian structure and semistability

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. The Lie algebra of G will be denoted by $\mathfrak{g}$. Fix a maximal compact subgroup $K \subset G$.

The group G has the adjoint action on $\mathfrak{g}$. Let h_0 be a K-invariant inner product on the complex vector space $\mathfrak{g}$. Let h_G be the unique Hermitian structure on G, invariant under the right-translation action of G on itself, with $h_G(e) = h_0$. Let ω_G be the $(1,1)$-form on G associated to h_G.

Proposition 2.1. *Let δ be the complex dimension of G. Then*

$$\mathrm{d}\omega_G^{\delta-1} = 0,$$

where ω_G is defined above.

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Proof. We will first reduce it to the case of semisimple groups. Let Z_G be the connected component of the center of G containing the identity element e. The Lie algebra of Z_G will be denoted by $\mathfrak{z}_{\mathfrak{g}}$. The Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}_{\mathfrak{g}}; \tag{1}$$

we note that $[\mathfrak{g}, \mathfrak{g}]$ is semisimple. Using h_0, any element $v \in \mathfrak{z}_{\mathfrak{g}}$ produces an element $\widetilde{v} \in [\mathfrak{g}, \mathfrak{g}]^*$ defined as follows:

$$\widetilde{v}(y) = h_0(y, v)$$

for all $y \in [\mathfrak{g}, \mathfrak{g}]$.

The action of G on $\mathfrak{g}$ preserves the decomposition in (1), and the action of G on $\mathfrak{z}_{\mathfrak{g}}$ is trivial. Since h_0 is K-invariant, these imply that $\widetilde{v}$ is left invariant by the action of K on $[\mathfrak{g}, \mathfrak{g}]^*$. We note that $[\mathfrak{g}, \mathfrak{g}]^*$ is identified with $[\mathfrak{g}, \mathfrak{g}]$ using the Killing form. There is no nonzero element of $[\mathfrak{g}, \mathfrak{g}]$ fixed by G because $[\mathfrak{g}, \mathfrak{g}]$ is semisimple; since K is Zariski dense in G, this implies that

$$[\mathfrak{g}, \mathfrak{g}]^K = 0.$$

In particular, $\widetilde{v} = 0$. Hence the decomposition in (1) is orthogonal with respect to h_0.

The natural projection

$$f : [G, G] \times Z_G \longrightarrow G$$

is a finite étale Galois covering. Since the decomposition in (1) is orthogonal, the 2-form $f^*\omega_G$ decomposes as

$$f^*\omega_G = p_1^*\omega_1 + p_2^*\omega_2, \tag{2}$$

where p_i is the projection of $[G, G] \times Z_G$ to the i-th factor, and ω_1 (respectively, ω_2) is the $(1, 1)$-form on $[G, G]$ (respectively, Z_G) associated to the right-translation invariant Hermitian metric obtained by translating $h_0|_{[\mathfrak{g},\mathfrak{g}]}$ (respectively, $h_0|_{\mathfrak{z}_{\mathfrak{g}}}$). From (2),

$$f^*\omega_G^{\delta-1} = \left(p_1^*\omega_1^{\delta_1-1}\right) \wedge p_2^*\omega_2^{\delta_2} + \left(p_1^*\omega_1^{\delta_1}\right) \wedge p_2^*\omega_2^{\delta_2-1},$$

where δ_1 and δ_2 are the complex dimensions of $[G, G]$ and Z_G respectively. Hence

$$f^*\,\mathrm{d}\omega_G^{\delta-1} = \mathrm{d}f^*\omega_G^{\delta-1} = \left(p_1^*\,\mathrm{d}\omega_1^{\delta_1-1}\right) \wedge p_2^*\omega_2^{\delta_2} + \left(p_1^*\omega_1^{\delta_1}\right) \wedge p_2^*\,\mathrm{d}\omega_2^{\delta_2-1}. \tag{3}$$

Since Z_G is abelian, it follows that $\mathrm{d}\omega_2 = 0$. Therefore, from (3) we conclude that $\mathrm{d}\omega_G^{\delta-1} = 0$ if $\mathrm{d}\omega_1^{\delta_1-1} = 0$. Therefore, it is enough to prove the proposition for G semisimple.

We assume that G is semisimple.

Since the inner product h_0 is K-invariant, the Hermitian structure h_G is invariant under the left-translation action of K on G. Therefore, the element

$$\left(\mathrm{d}\omega_G^{\delta-1}\right)(e) \in \wedge^{2\delta-1}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^* \tag{4}$$

is preserved by the action of K on $\wedge^{2\delta-1}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*$ constructed using the adjoint action of K on $\mathfrak{g}$.

The Killing form on $\mathfrak{g}$ produces a nondegenerate symmetric bilinear form on $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Using it, the K-module $\wedge^{2\delta-1}(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})^*$ gets identified with $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. There is no nonzero element of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ which is fixed by K because G is semisimple and K is Zariski dense in G. In particular, the K-invariant element $(\mathrm{d}\omega_G^{\delta-1})(e)$ in (4) vanishes. Since $\mathrm{d}\omega^{\delta-1}$ is invariant under the right-translation action of G on itself, and $(\mathrm{d}\omega_G^{\delta-1})(e) = 0$, we conclude that $\mathrm{d}\omega_G^{\delta-1} = 0$. □

Let

$$\Gamma \subset G$$

be a closed discrete subgroup such that the quotient manifold G/Γ is compact. The right-translation invariant Hermitian structure h_G on G descends to a Hermitian structure on G/Γ. This Hermitian structure on G/Γ will be denoted by $\widetilde{h}$. Let $\widetilde{\omega}$ be the $(1, 1)$-form on G/Γ defined by $\widetilde{h}$; so $\widetilde{\omega}$ pulls back to the form ω_G on G. From Proposition 2.1 we know that

$$\mathrm{d}\widetilde{\omega}^{\delta-1} = 0. \tag{5}$$

For a coherent analytic sheaf E on G/Γ, define the *degree* of E

$$\mathrm{degree}(E) := \int_{G/\Gamma} c_1\left(\det(E)\right) \wedge \widetilde{\omega}^{\delta-1};$$

from (5) it follows immediately that degree(E) is independent of the choice of the first Chern form for the (holomorphic) determinant line bundle $\det(E)$; see [5, Ch. V, § 6] for determinant bundle.

For any $g \in G$, let

$$\beta_g : M := G/\Gamma \longrightarrow G/\Gamma \tag{6}$$

be the left-translation automorphism defined by $x \longmapsto gx$. A coherent analytic sheaf E over G/Γ is called *invariant* if for each $g \in G$, the pulled back coherent analytic sheaf $\beta_g^* E$ is isomorphic to E. Note that an invariant coherent analytic sheaf is locally free.

Theorem 2.2. *Let E be an invariant holomorphic vector bundle on G/Γ. Then*

$$\text{degree}(E) = 0.$$

Proof. Since E is invariant, it admits a holomorphic connection [3, Theorem 3.1]. Any holomorphic connection on E induces a holomorphic connection on the determinant line bundle $\det(E) := \bigwedge^r E$, where r is the rank of E.

Let

$$D : \det(E) \longrightarrow \det(E) \otimes \Omega^{1,0}_{G/\Gamma}$$

be a holomorphic connection on $\det(E)$; see [1] for the definition of a holomorphic connection. Let

$$\bar\partial_{\det(E)} : \det(E) \longrightarrow \det(E) \otimes \Omega^{0,1}_{G/\Gamma}$$

be the Dolbeault operator defining the holomorphic structure on $\det(E)$. Then $D + \bar\partial_{\det(E)}$ is a connection on $\det(E)$. Let $\mathcal{K}(D + \bar\partial_{\det(E)})$ be the curvature of the connection $D + \bar\partial_{\det(E)}$. We note that

$$\mathcal{K}(D + \bar\partial_{\det(E)}) = (D + \bar\partial_{\det(E)})^2 = D^2$$

because the differential operator D is holomorphic, and $\bar\partial_{\det(E)}$ is integrable, meaning $\bar\partial^2_{\det(E)} = 0$. Therefore, $\mathcal{K}(D + \bar\partial_E)$ is a differential form of G/Γ of type $(2,0)$. (In fact, the form D^2, which is called the curvature of the holomorphic connection D, is holomorphic, but we do not need it.)

As $\mathcal{K}(D + \bar\partial_E)$ is of type $(2,0)$, and $\widetilde{\omega}$ is of type $(1,1)$,

$$\text{degree}\big(\det(E)\big) = \int_{G/\Gamma} \mathcal{K}(D + \bar\partial_E) \wedge \widetilde{\omega}^{\delta-1} = 0.$$

Since $c_1(E) = c_1(\det(E))$, and the degree depends only on the first Chern class due to Proposition 2.1, the theorem follows. □

A vector bundle E over G/Γ is called *semistable* if

$$\frac{\text{degree}(V)}{\text{rank}(V)} \leqslant \frac{\text{degree}(E)}{\text{rank}(E)}$$

for every coherent analytic subsheaf $V \subset E$ of positive rank.

Lemma 2.3. *Let E be a torsion-free coherent analytic sheaf on G/Γ. For any $g \in G$,*

$$\text{degree}(E) = \text{degree}\big(\beta_g^* E\big),$$

where β_g is the map in (6).

Proof. Since G is connected, the map β_g is homotopic to the identity map of G/Γ. Hence

$$c_1\big(\det\big(\beta_g^* E\big)\big) - c_1\big(\det(E)\big) = d\alpha,$$

where α is a smooth 1-form on G/Γ, and $c_1(\det(\beta_g^* E))$ and $c_1(\det(E))$ are first Chern forms. Now,

$$\begin{aligned}\text{degree}\big(\beta_g^* E\big) - \text{degree}(E) &= \int_{G/\Gamma} \big(c_1\big(\det\big(\beta_g^* E\big)\big) - c_1\big(\det(E)\big)\big) \wedge \widetilde{\omega}^{\delta-1} \\ &= \int_{G/\Gamma} (\mathrm{d}\alpha) \wedge \widetilde{\omega}^{\delta-1} = \int_{G/\Gamma} \alpha \wedge \mathrm{d}\widetilde{\omega}^{\delta-1} = 0\end{aligned}$$

by Proposition 2.1. □

Lemma 2.4. *Let E be an invariant holomorphic vector bundle on G/Γ. Then E is semistable.*

Proof. Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = E$$

be the unique Harder–Narasimhan filtration for E [4, p. 590, Theorem 3.2]. We recall that

$$\frac{\text{degree}(V_1)}{\text{rank}(V_1)} > \frac{\text{degree}(E)}{\text{rank}(E)} \tag{7}$$

if $V_1 \neq E$. From the uniqueness of V_1 and Lemma 2.3 it follows that for any $g \in G$ and any isomorphism of E with $\beta_g^* E$, the image of the composition

$$\beta_g^* V_1 \hookrightarrow \beta_g^* E \xrightarrow{\sim} E$$

coincides with V_1. This implies that V_1 is invariant. Therefore,

$$\text{degree}(V) = 0 = \text{degree}(E)$$

by Theorem 2.2. Now from (7) we conclude that $V_1 = E$. Hence E is semistable. □

Let H be any affine complex algebraic group. Let E_H be an invariant holomorphic principal H-bundle over G/P, which means that the principal H-bundle $\beta_g^* E_H$ is holomorphically isomorphic to E_H for all $g \in G$. From Lemma 2.4 we know that the adjoint vector bundle $\text{ad}(E_H)$ is semistable. If H is reductive then the semistability of $\text{ad}(E_H)$ implies that the principal H-bundle E_H is semistable; see [6] for the definition of semistable principal bundles.

It is a natural question to ask whether Lemma 2.4 remains valid if the reductive group G is replaced by some more general affine algebraic groups G_1. The first step would be to construct a suitable Hermitian structure on the compact quotient G_1/Γ, where Γ is a cocompact lattice in G_1. In order to be able to define semistability, the Hermitian structure on G_1/Γ should satisfy the Gauduchon condition. It may be noted that the Hermitian structure on G_1/Γ given by a right-translation invariant Hermitian structure on G_1 satisfies the Gauduchon condition (see [2, p. 74]). For a general compact complex manifold equipped with a Gauduchon metric, the degree of a line bundle with holomorphic connection need not be zero; but it remains valid if the base admits a Kähler metric [1, p. 196, Proposition 12]. The compact complex manifold G_1/Γ admits a Kähler metric if and only if G_1 is abelian.

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