Mathematical Problems in Mechanics

Nonlinear Saint-Venant compatibility conditions for nonlinearly elastic plates

Conditions non linéaires de compatibilité de Saint-Venant pour des plaques non linéairement élastiques

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1. The classical approach to nonlinear plate theory

Greek indices vary in \{1, 2\}, Latin indices vary in \{1, 2, 3\} (unless otherwise specified), and the convention summation with respect to repeated indices is used. Partial derivatives of the first, resp. second, order are denoted \( \partial_\alpha \) or \( \partial_i \), resp. \( \partial_\alpha \beta \)
or \(\delta_{ij}\). Vector fields are denoted by boldface letters. The space of all symmetric \(N \times N\) matrices is denoted \(\mathbb{S}^N\). Sets of symmetric matrix fields are denoted by special Roman capital letters.

A domain in \(\mathbb{R}^N\) is a bounded, open, and connected subset \(\Omega\) of \(\mathbb{R}^N\) with a Lipschitz-continuous boundary \(\Gamma\), the set \(\Omega\) being locally on the same side of \(\Gamma\).

To begin with, we briefly describe the classical Kirchhoff–von Kármán–Love model for a nonlinearly elastic plate (so named after Kirchhoff [7], von Kármán [6], and Love [8]), which constitutes the point of departure for the present work. This model has been fully justified from three-dimensional elasticity by means of Gamma-convergence theory by Friesecke, James and Müller [5].

Let \(\omega\) be a domain in \(\mathbb{R}^2\) and let \(\varepsilon > 0\). Assume that the set \(\overline{\omega} \times [-\varepsilon, \varepsilon]\) is the reference configuration of a nonlinearly elastic plate of thickness \(2\varepsilon\) made with a homogeneous and isotropic elastic material characterized by its two Lamé constants \(\lambda \geq 0\) and \(\mu > 0\) (the reference configuration is assumed to be a natural state). Let

\[
a_{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} \delta_{\alpha\beta} \delta_{\sigma\tau} + 2\mu (\delta_{\alpha\sigma} \delta_{\beta\tau} + \delta_{\alpha\tau} \delta_{\beta\sigma}),
\]

where \(\delta_{\alpha\beta}\) designates the Kronecker symbol, denote the components of the two-dimensional elasticity tensor of the plate, which thus satisfies

\[
a_{\alpha\beta\sigma\tau} t_{\sigma\tau} t_{\alpha\beta} \geq 4\mu \sum_{\alpha, \beta} |t_{\alpha\beta}|^2 \quad \text{for all} \quad (t_{\alpha\beta}) \in \mathbb{S}^2.
\]

The plate is subjected to applied forces, with resultants \(p_1 \in L^2(\omega)\) and \(q_\alpha \in L^2(\omega)\). Define the space

\[
V(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega).
\]

Then the associated displacement problem consists in finding a displacement vector field \(\xi = (\xi_i)\) of the set \(\overline{\omega}\) (the middle surface of the plate) that minimizes the functional \(J\) defined for each \(\eta = (\eta_i) \in V(\omega)\) by

\[
J(\eta) := \frac{1}{2} \int_\omega \left\{ \frac{\varepsilon^3}{4} a_{\alpha\beta\sigma\tau} (\partial_{\sigma} \eta_\tau + \partial_\tau \eta_\sigma + \partial_\sigma \eta_\tau \partial_\tau \eta_\sigma + \partial_\tau \eta_\sigma \partial_\sigma \eta_\tau + \partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_\beta \partial_\beta \eta_\alpha) + \frac{3}{2} a_{\alpha\beta\sigma\tau} \partial_\sigma \eta_\tau \partial_\alpha \eta_\beta \eta_\tau \right\} d\omega - L(\eta),
\]

where

\[
L(\eta) := \int_\omega p_1 \eta_1 d\omega - \int_\omega q_\alpha \partial_\alpha \eta_3 d\omega,
\]

over a closed subspace \(U(\omega)\) of \(V(\omega)\) that incorporates boundary conditions that are specific to the problem under consideration. For instance, if the plate is clamped over a portion of its lateral face,

\[
U(\omega) := \{ \eta = (\eta_i) \in V(\omega); \eta_1 = \partial_\alpha \eta_3 = 0 \text{ on } \gamma_0 \},
\]

where \(\gamma_0\) is a portion of \(\gamma := \partial \omega\) such that \(d\gamma\text{-meas } \gamma_0 > 0\). Then the corresponding minimization problem has at least one solution if the norms \(\|p_a\|_{L^2(\omega)}\) are small enough (Ciarlet and Destuynder [2]), or if \(\gamma = \emptyset\), in which case there is no longer any restriction on the magnitude of the norms \(\|p_a\|_{L^2(\omega)}\) (Rabier [10]). The case \(p_a = 0\) had been previously considered by Nečas and Neumann [9].

While the existence theory for the Dirichlet–Neumann problem \((0 < d\gamma\text{-meas } \gamma_0 < d\gamma\text{-meas } \gamma')\) and Dirichlet problem \((\gamma_0 = \emptyset)\) is thus well-established, little attention seems to have been given to the Neumann problem \((\gamma_0 = \emptyset)\), at least to the authors’ best knowledge.

In this respect, one of the outcome of our study will be the existence of a solution to the minimization problem when \(\gamma_0 = \emptyset\) (see Ciarlet and Mardare [3]). To this end, we will re-formulate this minimization problem in terms of the unknowns

\[
E_{\alpha\beta} := \frac{1}{2} (\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) \in L^2(\omega) \quad \text{and} \quad F_{\alpha\beta} := \partial_\alpha \eta_3 \in L^2(\omega), \quad \alpha, \beta = 1, 2,
\]

i.e., through an approach that extends to the non-quadratic minimization problem considered here the intrinsic approach applied by Ciarlet and Ciarlet Jr. [1] to the quadratic minimization problem of three-dimensional linearized elasticity. This is why our first aim is to introduce and analyze (see Sections 2 and 3) conditions that extend to the nonlinear Kirchhoff–Love plate theory the weak Saint-Venant compatibility conditions introduced in [1].

Complete proofs will be found in Ciarlet and Mardare [4].
2. Nonlinear Saint-Venant compatibility conditions

To begin with, we have the following nonlinear analog of Theorem 3.2 of [1]:

**Theorem 2.1** (Nonlinear Saint-Venant compatibility conditions). Let \( \omega \) be a simply-connected domain in \( \mathbb{R}^2 \) and let there be given two symmetric matrix fields \( \mathbf{E} = (E_{\alpha\beta}) \in L^2(\omega) := L^2(\omega; \mathbb{S}^2) \) and \( \mathbf{F} = (F_{\alpha\beta}) \in L^2(\omega) \) whose components satisfy the nonlinear Saint-Venant compatibility conditions:

\[
\begin{align*}
\partial_\tau E_{\alpha\beta} + \partial_\alpha E_{\tau\beta} - \partial_\alpha E_{\beta\tau} - \partial_\beta E_{\alpha\tau} = F_{\alpha\beta} - F_{\alpha\beta} & \quad \text{in } H^{-2}(\omega), \quad (1) \\
\partial_\alpha F_{\alpha\beta} = \partial_\beta F_{\alpha\beta} & \quad \text{in } H^{-1}(\omega). \quad (2)
\end{align*}
\]

Then there exists a vector field

\[ \mathbf{\eta} = (\eta_i) \in \mathbf{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega) \]

such that

\[
\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) = E_{\alpha\beta} \quad \text{in } L^2(\omega), \quad (3)
\]

\[
\partial_\alpha \eta_3 = F_{\alpha\beta} \quad \text{in } L^2(\omega). \quad (4)
\]

Besides, any other solution \( \tilde{\mathbf{\eta}} \) to Eqs. (3)-(4) is of the form

\[
\tilde{\mathbf{\eta}}(y) = \mathbf{\eta}(y) + a + b \mathbf{e} \wedge y - \eta_3(y) \mathbf{d} + (\mathbf{d} \cdot y) \mathbf{e} - \frac{1}{2}(\mathbf{d} \cdot \mathbf{d}) \mathbf{d} \quad \text{for almost all } y \in \omega,
\]

for some \( a \in \mathbb{R}^3, b \in \mathbb{R}, \) and \( \mathbf{d} \in \mathbb{R}^2, \) where \( (\mathbf{e}_i) := \delta_{i3}. \)

**Sketch of proof.** First, two successive applications of the weak **Poincaré lemma** (Theorem 3.1 in [1]) to Eqs. (2) show that there exists \( \eta_3 \in H^2(\omega) \) such that \( \partial_\alpha \eta_3 = F_{\alpha\beta} \) in \( L^2(\omega) \) (the assumption that \( \omega \) is simply-connected is used here). Second, let

\[
e_{\alpha\beta} := E_{\alpha\beta} - \frac{1}{2} \partial_\alpha \eta_3 \partial_\beta \eta_3 \in L^2(\omega).
\]

Combining the expressions of second-order partial derivatives such as \( \partial_\sigma \partial_\tau (\partial_\alpha \eta_3 \partial_\beta \eta_3) \) for smooth functions \( \eta_3 \) with the density of \( C^\infty(\partial) \) in \( H^1(\omega) \) and in \( H^2(\omega) \) and with the continuous injection of \( L^1(\omega) \) into \( H^{-2}(\omega) \) then eventually shows that the above functions \( e_{\alpha\beta} \) satisfy

\[
\partial_\sigma e_{\alpha\beta} + \partial_\alpha e_{\sigma\beta} - \partial_\alpha e_{\beta\sigma} - \partial_\beta e_{\alpha\sigma} = 0 \quad \text{in } H^{-2}(\omega),
\]

which are precisely the weak Saint-Venant compatibility conditions of Theorem 3.2 in [1] for \( N = 2 \). Hence this theorem shows that there exists a vector field \( \mathbf{\eta}_H = (\eta_\alpha) \in H^1(\omega) \) such that

\[
\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) = e_{\alpha\beta} = E_{\alpha\beta} - \frac{1}{2} \partial_\alpha \eta_3 \partial_\beta \eta_3 \quad \text{in } L^2(\omega)
\]

(the assumptions of simple-connectedness of \( \omega \) are again used here). The existence of a solution \( \mathbf{\eta} = (\eta_H, \eta_3) \in \mathbf{V}(\omega) \) to Eqs. (3)-(4) is thus established.

We next examine the question of **uniqueness**, for which only the assumption that \( \omega \) is **connected** (this assumption is contained in the assumption that \( \omega \) is simply-connected) is used. So, assume that \( \tilde{\mathbf{\eta}} = (\tilde{\eta_H}, \tilde{\eta}_3) \in \mathbf{V}(\omega) \) and \( \mathbf{\eta} = (\eta_H, \eta_3) \in \mathbf{V}(\omega) \) satisfy

\[
\frac{1}{2}(\partial_\alpha \tilde{\eta}_\beta + \partial_\beta \tilde{\eta}_\alpha + \partial_\alpha \tilde{\eta}_3 \partial_\beta \tilde{\eta}_3) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) \quad \text{in } L^2(\omega), \quad (6)
\]

\[
\partial_\alpha \tilde{\eta}_3 = \partial_\alpha \eta_3 \quad \text{in } L^2(\omega). \quad (7)
\]

It is then well known that, since \( \omega \) is connected, Eqs. (7) imply that there exist a constant \( a_3 \) and a vector \( \mathbf{d} \in \mathbb{R}^2 \) such that

\[ \tilde{\eta}_3 = \eta_3 + a_3 + \mathbf{d} \cdot \mathbf{d} \quad \text{a.e. in } \omega, \quad (8) \]

where \( \mathbf{id} \) denotes the identity mapping of the set \( \omega \). Using relation (8) in Eqs. (6) then implies that

\[
\frac{1}{2}(\partial_\alpha \tilde{\eta}_\beta + \partial_\beta \tilde{\eta}_\alpha) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) \quad \text{in } L^2(\omega), \quad (9)
\]
where
\[ \tilde{\eta}_H = (\tilde{\eta}_a) = \eta_H - \eta_3 d - \frac{1}{2} (d \cdot \text{id}) d. \]  \hspace{1cm} (10)

It is again well known that, since \( \omega \) is connected, relations (9) imply that there exist \( b \in \mathbb{R} \) and \( a_H \in \mathbb{R}^2 \) such that
\[ \tilde{\eta}_H = \tilde{\eta}_H + a_H + b e \wedge \text{id} \quad \text{a.e. in } \omega. \]  \hspace{1cm} (11)
Combining (8), (10), and (11), and letting \( a := (a_H, a_3) \) then yields (5).

Incidentally, Theorem 2.1 shows that, if a vector field \( \eta = (\eta_i) \in \mathbf{V}(\omega) \) satisfies
\[ \frac{1}{2} (\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) = 0 \quad \text{and} \quad \partial_\alpha \eta_3 = 0 \quad \text{a.e. in } \omega, \]
then there exist \( a \in \mathbb{R}^3, b \in \mathbb{R}, \) and \( d \in \mathbb{R}^2 \) such that \( \eta(y) = a + be \wedge y + (d \cdot y)e - \frac{1}{2} (d \cdot \text{id}) d \) for almost all \( y \in \omega. \)

One can show (see [4]) that the nonlinear Saint-Venant compatibility conditions (1)–(2) are also necessary. This means that, given any vector field \( \eta \in \mathbf{V}(\omega) \), the matrix fields \( E = (E_{\alpha\beta}) \in \mathbb{L}^2(\omega) \) and \( F = (F_{\alpha\beta}) \in \mathbb{L}^2(\omega) \) defined by Eqs. (3)–(4) necessarily satisfy the relations (1)–(2) in this case, the domain \( \omega \) need not be simply-connected.

Note that the nonlinear Saint-Venant compatibility conditions (1)–(2) reduce in fact to three relations only, e.g.,
\[ \partial_1 E_{22} + \partial_2 E_{11} - 2 \partial_1 E_{12} = (F_{12})^2 - F_{11} F_{22} \quad \text{in } H^{-2}(\omega), \]
\[ \partial_1 F_{22} = \partial_2 F_{11} \quad \text{in } H^{-1}(\omega). \]

Finally, note that Eqs. (3)–(4) can be also written in matrix form as
\[ \nabla_3 \eta_H + \frac{1}{2} \nabla \eta_3 \nabla \eta_3^T = E \quad \text{and} \quad \nabla \eta_3 = F \quad \text{in } \mathbb{L}^2(\omega), \]
where \( (\nabla \eta_H)_{\alpha\beta} := \frac{1}{2} (\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) \) and \( \nabla \eta_3 := (\partial_\alpha \eta_3) \), so that \( \nabla \eta_3 \nabla \eta_3^T = (\partial_\alpha \eta_3 \partial_\beta \eta_3). \)

We now introduce a closed subspace \( \mathbf{V}^0(\omega) \) of \( \mathbf{V}(\omega) \) in which the uniqueness of a vector field \( \eta \) satisfying Eqs. (3) and (4) is guaranteed.

**Theorem 2.2.** Let \( \omega \) be a simply-connected domain in \( \mathbb{R}^2 \). Define the space
\[ \mathcal{E}(\omega) := \{ (E, F) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega); \partial_\sigma \tau E_{\alpha\beta} + \partial_\alpha \beta E_{\sigma\tau} - \partial_\alpha \sigma E_{\beta\tau} - \partial_\beta \tau E_{\alpha\sigma} = F_{\alpha\sigma} F_{\beta\tau} - F_{\alpha\beta} F_{\sigma\tau} \quad \text{in } H^{-2}(\omega), \]
\[ \partial_\sigma \alpha \beta = \partial_\beta \alpha \sigma \quad \text{in } H^{-1}(\omega) \}. \]  \hspace{1cm} (12)

Then, given any \( (E, F) \in \mathcal{E}(\omega) \), there exists a unique vector field
\[ \eta \in \mathbf{V}^0(\omega) := \{ \eta = (\eta_i) \in \mathbf{V}(\omega), \int_\omega \eta \, d\omega = 0, \int_\omega \partial_\alpha \eta_3 \, d\omega = 0, \int_\omega (\partial_1 \eta_2 - \partial_2 \eta_1) \, d\omega = 0 \} \]  \hspace{1cm} (13)
that satisfies Eqs. (3)–(4).

**Sketch of proof.** By Theorem 2.1, there exists \( \eta = (\eta_H, \eta_3) \in \mathbf{V}(\omega) \) such that Eqs. (3)–(4) are satisfied; besides, for any \( a \in \mathbb{R}^3, b \in \mathbb{R}, \) and \( d \in \mathbb{R}^2, \)
\[ \eta^0 := \eta + a + be \wedge - \eta_3 d + (d \cdot \text{id}) e - \frac{1}{2} (d \cdot \text{id}) d \]  \hspace{1cm} (14)
is also a solution to Eqs. (3)–(4). Let \( d := (-f_\alpha \partial_\alpha \eta_3) \) be a solution to Eqs. (3)–(4), so that \( \int_\omega \partial_\alpha \eta_3 \, d\omega = 0 \); it is then easily seen that there exist \( a \in \mathbb{R}^3 \) and \( b \in \mathbb{R} \) such that the corresponding vector field \( \eta^0 \) (as defined in (14)) belongs to the space \( \mathbf{V}^0(\omega) \).

To show that such a vector field \( \eta^0 \) is unique, assume that \( \tilde{\eta}^0 \in \mathbf{V}^0(\omega) \) also satisfies Eqs. (3)–(4), so that \( \tilde{\eta}^0 \) is necessarily of the form
\[ \tilde{\eta}^0 = \eta^0 + a + be \wedge - \eta_3 d + (d \cdot \text{id}) e - \frac{1}{2} (d \cdot \text{id}) d \]
for some \( a = (a_H, a_3) \in \mathbb{R}^3, b \in \mathbb{R}, \) and \( d \in \mathbb{R}^2. \) It is then easily seen, first that \( d = 0, \) then that \( a_3 = 0, b = 0, \) and \( a_H = 0. \)
3. Continuity of the mapping \((E, F) \in \mathbb{E}(\omega) \to \eta \in V^0(\omega)\)

We have the following nonlinear analog of Theorem 4.1 of [1]. The spaces \(\mathbb{E}(\omega)\) and \(V^0(\omega)\) are those defined in (12) and (13).

**Theorem 3.1.** Let \(\omega\) be a simply-connected domain, and let

\[
\Phi : \mathbb{E}(\omega) \to V^0(\omega)
\]

be the nonlinear bijection defined for each \((E, F) \in \mathbb{E}(\omega)\) by \(\Phi(E, F) := \eta\), where \(\eta\) is the unique element in the space \(V^0(\omega)\) that satisfies Eqs. (3)-(4) (Theorem 2.2). Then there exists a constant \(C\) such that

\[
\left\| \Phi(E, F) \right\|_{H^1(\omega) \times H^1(\omega) \times H^2(\omega)} \leq C \left( \left\| E \right\|_{L^2(\omega)} + \left\| F \right\|_{L^2(\omega)} + \left\| F \right\|_{L^2(\omega)}^2 \right) \text{ for all } (E, F) \in \mathbb{E}(\omega).
\]

Besides, the set \(\mathbb{E}(\omega)\) is sequentially weakly closed in \(L^2(\omega) \times L^2(\omega)\), and \(\Phi\) maps weakly convergent sequences in \(\mathbb{E}(\omega)\) endowed with the topology of \(L^2(\omega) \times L^2(\omega)\) into strongly convergent sequences in \(V^0(\omega)\) endowed with the topology of \(L^2(\omega) \times L^2(\omega) \times H^1(\omega)\).

**Sketch of proof.** That the nonlinear mapping \(\Phi\) is a bijection from \(\mathbb{E}(\omega)\) onto \(V^0(\omega)\) follows from necessity of the nonlinear Saint-Venant compatibility conditions, and from their sufficiency (established in Theorem 2.2). Besides, for each \(\eta = (\eta_H, \eta_3) \in V^0(\omega)\),

\[
\Phi^{-1}(\eta) = \left( \nabla \cdot \eta_H + \frac{1}{2} \nabla \eta_3 \nabla \eta_3^T, \nabla^2 \eta_3 \right).
\]

Given any \(\eta = (\eta_H, \eta_3) \in V^0(\omega)\), the function \(\eta_3 \in H^2(\omega)\) satisfies \(\int_\omega \eta_3 \, d\omega = 0\). Hence the Poincaré-Wirtinger inequality implies that there exists a constant \(C_1\) such that

\[
\left\| \eta_3 \right\|_{H^2(\omega)} \leq C_1 \left\| \nabla^2 \eta_3 \right\|_{L^2(\omega)} \text{ for all } \eta \in V^0(\omega).
\]

Writing \(\nabla \cdot \eta_H = (\nabla \cdot \eta_H + \frac{1}{2} \nabla \eta_3 \nabla \eta_3^T) - \frac{1}{2} \nabla \eta_3 (\nabla \eta_3)^T\), we then infer from the classical two-dimensional Korn’s inequality that there exists a constant \(C_2\) such that

\[
\left\| \eta_H \right\|_{H^1(\omega) \times H^1(\omega)} \leq C_2 \left\| \nabla \cdot \eta_H + \frac{1}{2} \nabla \eta_3 \nabla \eta_3^T \right\|_{L^2(\omega)} + \left\| \nabla \eta_3 (\nabla \eta_3)^T \right\|_{L^2(\omega)} \text{ for all } \eta \in V^0(\omega).
\]

Given any \(\eta = (\eta_H, \eta_3) \in V^0(\omega)\), the vector field \(\nabla \eta_3 \in H^1(\omega) \times H^1(\omega)\) satisfies \(\int_\omega \nabla \eta_3 \, d\omega = 0\); besides, the continuous injection \(H^1(\omega) \hookrightarrow L^4(\omega)\) holds. Hence there exist constants \(C_3\) and \(C_4\) such that

\[
\left\| \nabla \eta_3 \right\|_{L^4(\omega)} \leq C_3 \left\| \nabla \eta_3 \right\|_{H^1(\omega)} \leq C_4 \left\| \eta_3 \right\|_{H^2(\omega)} \leq C_1 C_4 \left\| \nabla^2 \eta_3 \right\|_{L^2(\omega)} \text{ for all } \eta \in V^0(\omega).
\]

Since, finally, there exists a constant \(C_5\) such that

\[
\left\| \nabla \eta_3 (\nabla \eta_3)^T \right\|_{L^2(\omega)} \leq C_5 \left( \left\| \nabla \eta_3 \right\|_{L^4(\omega)} \right)^2 \text{ for all } \eta \in V^0(\omega),
\]

inequality (15) follows by combining the above inequalities.

In what follows, \(\to\), resp. \(\rightharpoonup\), denotes strong, resp. weak, convergence. Let \((E^k, F^k) \in \mathbb{E}(\omega), k \geq 1\), and \((E, F) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)\) be such that

\[
(E^k, F^k) \rightharpoonup (E, F) \text{ in } \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega) \text{ as } k \to \infty.
\]

By inequality (15), the sequence \((\eta^k)_{k=1}^\infty\), where \(\eta^k := \Phi(E^k, F^k) \in V^0(\omega)\), is then bounded in \(V^0(\omega)\). Since \(V^0(\omega)\) is reflexive (as a closed subspace of \(H^1(\omega) \times H^1(\omega) \times H^2(\omega)\)), there exists a subsequence \((\eta^k)_{k=1}^\infty\) and \(\eta \in V^0(\omega)\) such that

\[
\eta^k \rightharpoonup \eta \text{ in } H^1(\omega) \times H^1(\omega) \times H^2(\omega) \text{ and } \eta^k \to \eta \text{ in } L^2(\omega) \times L^2(\omega) \times H^1(\omega).
\]

Hence \(E_{\alpha \beta}^k = \partial_{\alpha \beta} \eta^k \to \partial_{\alpha \beta} \eta \) in \(L^2(\omega)\), which shows that \(F_{\alpha \beta}^k = \partial_{\alpha \beta} \eta^k \eta_3\) (uniqueness of the weak limit). Furthermore \(\eta_3^k \to \eta_3\) in \(H^1(\omega)\) implies \(\partial_{\alpha} \eta_3^k \to \partial_{\alpha} \eta_3\) in \(L^2(\omega)\), so that \(\partial_{\alpha} \eta_3^k \partial_{\beta} \eta_3^k \to \partial_{\alpha} \eta_3 \partial_{\beta} \eta_3\) in \(L^1(\omega)\). Since \(\frac{1}{2} (\partial_{\alpha} \eta_\beta + \partial_{\beta} \eta_\alpha) \to \frac{1}{2} (\partial_{\alpha} \eta_\beta + \partial_{\beta} \eta_\alpha)\) in \(L^2(\omega)\), it follows that, for each \(\varphi \in \mathbb{D}(\omega)\),

\[
\int_\omega E_{\alpha \beta} \varphi \, d\omega \to \int_\omega \frac{1}{2} (\partial_{\alpha} \eta_\beta + \partial_{\beta} \eta_\alpha + \partial_{\alpha} \eta_3 \partial_{\beta} \eta_3) \varphi \, d\omega,
\]

which shows that \(E_{\alpha \beta} = \frac{1}{2} (\partial_{\alpha} \eta_\beta + \partial_{\beta} \eta_\alpha + \partial_{\alpha} \eta_3 \partial_{\beta} \eta_3)\). Consequently, \((E, F) \in \mathbb{E}(\omega)\) since \(\eta \in V^0(\omega)\). Therefore \(\mathbb{E}(\omega)\) is sequentially weakly closed.
Finally, the uniqueness of the limit implies that the whole sequence $\eta^k$ strongly converges to $\eta$ in $L^2(\omega) \times L^2(\omega) \times H^1(\omega)$. □

Note that, when equivalently expressed in terms of the vector fields $\eta \in V^0(\omega)$ (instead of the matrix fields $(E, F)$ in the space $E(\omega)$ of (12), inequality (15) provides an instance of a nonlinear Korn's inequality.

In [3], Theorem 3.1 will be put to use for establishing the existence of a minimizer over the space $E(\omega)$ of the functional $J : E(\omega) \to \mathbb{R}$ defined for each $(E, F) \in E(\omega)$ by

$$J(E, F) := \frac{1}{2} \int_\omega \left\{ \varepsilon_{a\beta\sigma\tau} E_{\sigma\tau} E_{a\beta} + \frac{\varepsilon^3}{3} a_{a\beta\sigma\tau} F_{\sigma\tau} F_{a\beta} \right\} \, d\omega - L(\Phi(E, F)),$$  \hfill (20)

when $p_H = 0$ (if $p_H \neq 0$, a vector field in $\mathbb{R}^2$ must be introduced as an extra variable; cf. [3]), thereby justifying the intrinsic approach for the Neumann problem described in Section 1. Besides, the convexity of the integrand in the functional $J$ of (20) with respect to its arguments $E = (E_{a\beta}) \in L^2(\omega)$ and $F = (F_{a\beta}) \in L^2(\omega)$ will lay the ground for defining a notion of polyconvexity adapted to the Kirchhoff–von Kármán–Love theory of nonlinearly elastic plates.

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References