C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1297–1302

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical Problems in Mechanics

Nonlinear Saint-Venant compatibility conditions for nonlinearly elastic plates

Conditions non linéaires de compatibilité de Saint-Venant pour des plaques non linéairement élastiques

Philippe G. Ciarlet [a], Sorin Mardare [b]

[a] *Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*
[b] *Laboratoire de mathématiques Raphaël-Salem, université de Rouen, avenue de l'université, 76801 Saint-Etienne-du-Rouvray, France*

ARTICLE INFO

Article history:
Received and accepted 20 October 2011
Available online 9 November 2011

Presented by Philippe G. Ciarlet

ABSTRACT

Let ω be a simply-connected planar domain. We give necessary and sufficient nonlinear compatibility conditions of Saint-Venant type guaranteeing that, given two 2×2 symmetric matrix fields $(E_{\alpha\beta})$ and $(F_{\alpha\beta})$ with components in $L^2(\omega)$, there exists a vector field $(\eta_i)_{i=1}^3$ with components $\eta_1, \eta_2 \in H^1(\omega)$ and $\eta_3 \in H^2(\omega)$ such that $\frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) = E_{\alpha\beta}$ and $\partial_{\alpha\beta}\eta_3 = F_{\alpha\beta}$ in ω for $\alpha, \beta = 1, 2$, the left-hand sides of these equations arising naturally in nonlinearly elastic plate theory. Such a vector field $\boldsymbol{\eta} = (\eta_i)$ being uniquely defined if it belongs to a particular closed subspace $\boldsymbol{V}^0(\omega)$ of $H^1(\omega)\times H^1(\omega)\times H^2(\omega)$, we study the continuity properties of the nonlinear mapping $(\boldsymbol{E}, \boldsymbol{F}) \in (L^2(\omega))^4 \times (L^2(\omega))^4 \to \boldsymbol{\eta} \in \boldsymbol{V}^0(\omega)$ defined in this fashion.

RÉSUMÉ

Soit ω un domaine plan simplement connexe. On donne des conditions non linéaires de compatibilité du type de Saint-Venant, nécessaires et suffisantes pour que, étant donné deux champs $(E_{\alpha\beta})$ et $(F_{\alpha\beta})$ de matrices symétriques dont les éléments sont dans $L^2(\omega)$, il existe un champ de vecteurs $(\eta_i)_{i=1}^3$ avec des composantes $\eta_1, \eta_2 \in H^1(\omega)$ et $\eta_3 \in H^2(\omega)$ tel que $\frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) = E_{\alpha\beta}$ et $\partial_{\alpha\beta}\eta_3 = F_{\alpha\beta}$ dans ω pour $\alpha, \beta = 1, 2$, les membres de gauche de ces équations apparaissant naturellement dans la théorie des plaques non linéairement élastiques. Un tel champ de vecteurs $\boldsymbol{\eta} = (\eta_i)$ étant défini de façon unique s'il appartient à un sous-espace fermé $\boldsymbol{V}^0(\omega)$ particulier de $H^1(\omega)\times H^1(\omega)\times H^2(\omega)$, on étudie les propriétés de continuité de l'application non linéaire $(\boldsymbol{E}, \boldsymbol{F}) \in (L^2(\omega))^4 \times (L^2(\omega))^4 \to \boldsymbol{\eta} \in \boldsymbol{V}^0(\omega)$ définie de cette façon.

1. The classical approach to nonlinear plate theory

Greek indices vary in $\{1, 2\}$, Latin indices vary in $\{1, 2, 3\}$ (unless otherwise specified), and the convention summation with respect to repeated indices is used. Partial derivatives of the first, resp. second, order are denoted ∂_α or ∂_i, resp. $\partial_{\alpha\beta}$

E-mail addresses: mapgc@cityu.edu.hk (P.G. Ciarlet), sorin.mardare@univ-rouen.fr (S. Mardare).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.10.019

or ∂_{ij}. Vector fields are denoted by boldface letters. The space of all symmetric $N \times N$ matrices is denoted $\mathbb{S}^N$. Sets of symmetric matrix fields are denoted by special Roman capital letters.

A *domain* in $\mathbb{R}^N$ is a bounded, open, and connected subset Ω of $\mathbb{R}^N$ with a Lipschitz-continuous boundary Γ, the set Ω being locally on the same side of Γ.

To begin with, we briefly describe the classical *Kirchhoff–von Kármán–Love model* for a *nonlinearly elastic plate* (so named after Kirchhoff [7], von Kármán [6], and Love [8]), which constitutes the point of departure for the present work. This model has been fully justified from three-dimensional elasticity by means of Gamma-convergence theory by Friesecke, James and Müller [5].

Let ω be a domain in $\mathbb{R}^2$ and let $\varepsilon > 0$. Assume that the set $\overline{\omega} \times [-\varepsilon, \varepsilon]$ is the reference configuration of a *nonlinearly elastic plate* of thickness 2ε made with a homogeneous and isotropic elastic material characterized by its two Lamé constants $\lambda \geqslant 0$ and $\mu > 0$ (the reference configuration is assumed to be a natural state). Let

$$a_{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu}\delta_{\alpha\beta}\delta_{\sigma\tau} + 2\mu(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma}),$$

where $\delta_{\alpha\beta}$ designates the Kronecker symbol, denote the components of the *two-dimensional elasticity tensor* of the plate, which thus satisfies

$$a_{\alpha\beta\sigma\tau}t_{\sigma\tau}t_{\alpha\beta} \geqslant 4\mu \sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \quad \text{for all } (t_{\alpha\beta}) \in \mathbb{S}^2.$$

The plate is subjected to *applied forces*, with resultants $p_i \in L^2(\omega)$ and $q_\alpha \in L^2(\omega)$. Define the space

$$\boldsymbol{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega).$$

Then the associated displacement problem consists in finding a *displacement vector field* $\boldsymbol{\zeta} = (\zeta_i)$ of the set $\overline{\omega}$ (the middle surface of the plate) that minimizes the functional J defined for each $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)$ by

$$J(\boldsymbol{\eta}) := \frac{1}{2}\int_\omega \left\{ \frac{\varepsilon}{4} a_{\alpha\beta\sigma\tau}(\partial_\sigma\eta_\tau + \partial_\tau\eta_\sigma + \partial_\sigma\eta_3\partial_\tau\eta_3)(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) + \frac{\varepsilon^3}{3} a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\eta_3\partial_{\alpha\beta}\eta_3 \right\} \mathrm{d}\omega - L(\boldsymbol{\eta}),$$

where

$$L(\boldsymbol{\eta}) := \int_\omega p_i\eta_i \,\mathrm{d}\omega - \int_\omega q_\alpha\partial_\alpha\eta_3 \,\mathrm{d}\omega,$$

over a closed subspace $\boldsymbol{U}(\omega)$ of $\boldsymbol{V}(\omega)$ that incorporates boundary conditions that are specific to the problem under consideration. For instance, if the plate is clamped over a portion of its lateral face,

$$\boldsymbol{U}(\omega) := \left\{ \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega);\ \eta_i = \partial_\alpha\eta_3 = 0 \text{ on } \gamma_0 \right\},$$

where γ_0 is a portion of $\gamma := \partial\omega$ such that $\mathrm{d}\gamma$-meas $\gamma_0 > 0$. Then the corresponding minimization problem has at least one solution if the norms $\|p_\alpha\|_{L^2(\omega)}$ are small enough (Ciarlet and Destuynder [2]), or if $\gamma = \gamma_0$, in which case there is no longer any restriction on the magnitude of the norms $\|p_\alpha\|_{L^2(\omega)}$ (Rabier [10]). The case $p_\alpha = 0$ had been previously considered by Nečas and Naumann [9].

While the existence theory for the *Dirichlet–Neumann problem* ($0 < \mathrm{d}\gamma$-meas $\gamma_0 < \mathrm{d}\gamma$-meas γ) and *Dirichlet problem* ($\gamma_0 = \gamma$) is thus well-established, little attention seems to have been given to the *Neumann problem* ($\gamma_0 = \varnothing$), at least to the authors' best knowledge.

In this respect, one of the outcome of our study will be the *existence of a solution to the minimization problem when* $\gamma_0 = \varnothing$ (see Ciarlet and Mardare [3]). To this end, we will re-formulate this minimization problem in terms of the unknowns

$$E_{\alpha\beta} := \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \partial_\alpha\eta_3\partial_\beta\eta_3) \in L^2(\omega) \quad \text{and} \quad F_{\alpha\beta} := \partial_{\alpha\beta}\eta_3 \in L^2(\omega), \quad \alpha, \beta = 1, 2,$$

i.e., through an approach that extends to the *non-quadratic* minimization problem considered here the *intrinsic approach* applied by Ciarlet and Ciarlet Jr. [1] to the quadratic minimization problem of three-dimensional linearized elasticity. This is why our first aim is to introduce and analyze (see Sections 2 and 3) conditions that extend to the nonlinear Kirchhoff–Love plate theory the *weak Saint-Venant compatibility conditions* introduced in [1].

Complete proofs will be found in Ciarlet and Mardare [4].

2. Nonlinear Saint-Venant compatibility conditions

To begin with, we have the following nonlinear analog of Theorem 3.2 of [1]:

Theorem 2.1 *(Nonlinear Saint-Venant compatibility conditions). Let ω be a simply-connected domain in $\mathbb{R}^2$ and let there be given two symmetric matrix fields $\boldsymbol{E} = (E_{\alpha\beta}) \in \mathbb{L}^2(\omega) := L^2(\omega; \mathbb{S}^2)$ and $\boldsymbol{F} = (F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ whose components satisfy the* nonlinear Saint-Venant compatibility conditions:

$$\partial_{\sigma\tau} E_{\alpha\beta} + \partial_{\alpha\beta} E_{\sigma\tau} - \partial_{\alpha\sigma} E_{\beta\tau} - \partial_{\beta\tau} E_{\alpha\sigma} = F_{\alpha\sigma} F_{\beta\tau} - F_{\alpha\beta} F_{\sigma\tau} \quad \text{in } H^{-2}(\omega), \tag{1}$$

$$\partial_\sigma F_{\alpha\beta} = \partial_\beta F_{\alpha\sigma} \quad \text{in } H^{-1}(\omega). \tag{2}$$

Then there exists a vector field

$$\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega) := H^1(\omega) \times H^1(\omega) \times H^2(\omega)$$

such that

$$\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) = E_{\alpha\beta} \quad \text{in } L^2(\omega), \tag{3}$$

$$\partial_{\alpha\beta} \eta_3 = F_{\alpha\beta} \quad \text{in } L^2(\omega). \tag{4}$$

Besides, any other solution $\tilde{\boldsymbol{\eta}}$ to Eqs. (3)–(4) *is of the form*

$$\tilde{\boldsymbol{\eta}}(y) = \boldsymbol{\eta}(y) + \boldsymbol{a} + b\boldsymbol{e} \wedge \boldsymbol{y} - \eta_3(y)\boldsymbol{d} + (\boldsymbol{d} \cdot \boldsymbol{y})\boldsymbol{e} - \frac{1}{2}(\boldsymbol{d} \cdot \boldsymbol{y})\boldsymbol{d} \quad \text{for almost all } y \in \omega, \tag{5}$$

for some $\boldsymbol{a} \in \mathbb{R}^3$, $b \in \mathbb{R}$, and $\boldsymbol{d} \in \mathbb{R}^2$, where $(\boldsymbol{e})_i := \delta_{i3}$.

Sketch of proof. First, two successive applications of the *weak Poincaré lemma* (Theorem 3.1 in [1]) to Eqs. (2) show that there exists $\eta_3 \in H^2(\omega)$ such that $\partial_{\alpha\beta}\eta_3 = F_{\alpha\beta}$ in $L^2(\omega)$ (the assumption that ω is simply-connected is used here). Second, let

$$e_{\alpha\beta} := E_{\alpha\beta} - \frac{1}{2}\partial_\alpha \eta_3 \partial_\beta \eta_3 \in L^2(\omega).$$

Combining the expressions of second-order partial derivatives such as $\partial_{\sigma\tau}(\partial_\alpha \eta_3 \partial_\beta \eta_3)$ for smooth functions η_3 with the density of $\mathcal{C}^\infty(\overline{\omega})$ in $H^1(\omega)$ and in $H^2(\omega)$ and with the continuous injection of $L^1(\omega)$ into $H^{-2}(\omega)$ then eventually shows that the above functions $e_{\alpha\beta}$ satisfy

$$\partial_{\sigma\tau} e_{\alpha\beta} + \partial_{\alpha\beta} e_{\sigma\tau} - \partial_{\alpha\sigma} e_{\beta\tau} - \partial_{\beta\tau} e_{\alpha\sigma} = 0 \quad \text{in } H^{-2}(\omega),$$

which are precisely the *weak Saint-Venant compatibility conditions* of Theorem 3.2 in [1] for $N = 2$. Hence this theorem shows that there exists a vector field $\boldsymbol{\eta}_H = (\eta_\alpha) \in \boldsymbol{H}^1(\omega)$ such that

$$\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) = e_{\alpha\beta} = E_{\alpha\beta} - \frac{1}{2}\partial_\alpha \eta_3 \partial_\beta \eta_3 \quad \text{in } L^2(\omega)$$

(the assumptions of simple-connectedness of ω is again used here). The *existence* of a solution $\boldsymbol{\eta} = (\boldsymbol{\eta}_H, \eta_3) \in \boldsymbol{V}(\omega)$ to Eqs. (3)–(4) is thus established.

We next examine the question of *uniqueness*, for which only the assumption that ω is *connected* (this assumption is contained in the assumption that ω is simply-connected) is used. So, assume that $\tilde{\boldsymbol{\eta}} = (\tilde{\boldsymbol{\eta}}_H, \tilde{\eta}_3) \in \boldsymbol{V}(\omega)$ and $\boldsymbol{\eta} = (\boldsymbol{\eta}_H, \eta_3) \in \boldsymbol{V}(\omega)$ satisfy

$$\frac{1}{2}(\partial_\alpha \tilde{\eta}_\beta + \partial_\beta \tilde{\eta}_\alpha + \partial_\alpha \tilde{\eta}_3 \partial_\beta \tilde{\eta}_3) = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) \quad \text{in } L^2(\omega), \tag{6}$$

$$\partial_{\alpha\beta} \tilde{\eta}_3 = \partial_{\alpha\beta} \eta_3 \quad \text{in } L^2(\omega). \tag{7}$$

It is then well known that, since ω is connected, Eqs. (7) imply that there exist a constant a_3 and a vector $\boldsymbol{d} \in \mathbb{R}^2$ such that

$$\tilde{\eta}_3 = \eta_3 + a_3 + \boldsymbol{d} \cdot \mathbf{id} \quad \text{a.e. in } \omega, \tag{8}$$

where **id** denotes the identity mapping of the set ω. Using relation (8) in Eqs. (6) then implies that

$$\frac{1}{2}(\partial_\alpha \tilde{\eta}_\beta + \partial_\beta \tilde{\eta}_\alpha) = \frac{1}{2}(\partial_\alpha \hat{\eta}_\beta + \partial_\beta \hat{\eta}_\alpha) \quad \text{in } L^2(\omega), \tag{9}$$

where

$$\hat{\boldsymbol{\eta}}_H = (\hat{\eta}_\alpha) = \boldsymbol{\eta}_H - \eta_3 \boldsymbol{d} - \frac{1}{2}(\boldsymbol{d} \cdot \mathbf{id})\boldsymbol{d}. \tag{10}$$

It is again well known that, since ω is connected, relations (9) imply that there exist $b \in \mathbb{R}$ and $\boldsymbol{a}_H \in \mathbb{R}^2$ such that

$$\tilde{\boldsymbol{\eta}}_H = \hat{\boldsymbol{\eta}}_H + \boldsymbol{a}_H + b\boldsymbol{e} \wedge \mathbf{id} \quad \text{a.e. in } \omega. \tag{11}$$

Combining (8), (10), and (11), and letting $\boldsymbol{a} := (\boldsymbol{a}_H, a_3)$ then yields (5). □

Incidentally, Theorem 2.1 shows that, if a vector field $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega)$ satisfies

$$\frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) = 0 \quad \text{and} \quad \partial_{\alpha\beta}\eta_3 = 0 \quad \text{a.e. in } \omega,$$

then there exist $\boldsymbol{a} \in \mathbb{R}^3$, $b \in \mathbb{R}$, and $\boldsymbol{d} \in \mathbb{R}^2$ such that $\boldsymbol{\eta}(y) = \boldsymbol{a} + b\boldsymbol{e} \wedge \boldsymbol{y} + (\boldsymbol{d} \cdot \boldsymbol{y})\boldsymbol{e} - \frac{1}{2}(\boldsymbol{d} \cdot \boldsymbol{y})\boldsymbol{d}$ for almost all $y \in \omega$.

One can show (see [4]) that the nonlinear Saint-Venant compatibility conditions (1)–(2) are also *necessary*. This means that, given any vector field $\boldsymbol{\eta} \in \boldsymbol{V}(\omega)$, the matrix fields $\boldsymbol{E} = (E_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ and $\boldsymbol{F} = (F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ *defined* by Eqs. (3)–(4) necessarily satisfy the relations (1)–(2) (in this case, the domain ω need not be simply-connected).

Note that the nonlinear Saint-Venant compatibility conditions (1)–(2) reduce in fact to *three* relations only, e.g.,

$$\partial_{11}E_{22} + \partial_{22}E_{11} - 2\partial_{12}E_{12} = (F_{12})^2 - F_{11}F_{22} \quad \text{in } H^{-2}(\omega),$$
$$\partial_1 F_{\alpha 2} = \partial_2 F_{\alpha 1} \quad \text{in } H^{-1}(\omega).$$

Finally, note that Eqs. (3)–(4) can be also written in *matrix form* as

$$\nabla_s \boldsymbol{\eta}_H + \frac{1}{2}\nabla\eta_3 \nabla\eta_3^T = \boldsymbol{E} \quad \text{and} \quad \nabla^2 \eta_3 = \boldsymbol{F} \quad \text{in } \mathbb{L}^2(\omega),$$

where $(\nabla_s \boldsymbol{\eta}_H)_{\alpha\beta} := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha)$ and $\nabla\eta_3 := (\partial_\alpha \eta_3)$, so that $\nabla\eta_3 \nabla\eta_3^T = (\partial_{\alpha\beta}\eta_3)$.

We now introduce a closed subspace $\boldsymbol{V}^0(\omega)$ of $\boldsymbol{V}(\omega)$ in which the *uniqueness* of a vector field $\boldsymbol{\eta}$ satisfying Eqs. (3) and (4) is guaranteed.

Theorem 2.2. *Let ω be a simply-connected domain in $\mathbb{R}^2$. Define the space*

$$\begin{aligned}\mathbb{E}(\omega) := \big\{&(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega);\ \partial_{\sigma\tau}E_{\alpha\beta} + \partial_{\alpha\beta}E_{\sigma\tau} - \partial_{\alpha\sigma}E_{\beta\tau} - \partial_{\beta\tau}E_{\alpha\sigma} = F_{\alpha\sigma}F_{\beta\tau} - F_{\alpha\beta}F_{\sigma\tau} \text{ in } H^{-2}(\omega),\\ &\partial_\sigma F_{\alpha\beta} = \partial_\beta F_{\alpha\sigma} \text{ in } H^{-1}(\omega)\big\}.\end{aligned} \tag{12}$$

Then, given any $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega)$, there exists a unique vector field

$$\boldsymbol{\eta} \in \boldsymbol{V}^0(\omega) := \left\{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega),\ \int_\omega \boldsymbol{\eta}\,\mathrm{d}\omega = \mathbf{0},\ \int_\omega \partial_\alpha \eta_3\,\mathrm{d}\omega = 0,\ \int_\omega (\partial_1\eta_2 - \partial_2\eta_1)\,\mathrm{d}\omega = 0\right\} \tag{13}$$

that satisfies Eqs. (3)–(4).

Sketch of proof. By Theorem 2.1, there exists $\boldsymbol{\eta} = (\boldsymbol{\eta}_H, \eta_3) \in \boldsymbol{V}(\omega)$ such that Eqs. (3)–(4) are satisfied; besides, for any $\boldsymbol{a} \in \mathbb{R}^3$, $b \in \mathbb{R}$, and $\mathbf{d} \in \mathbb{R}^2$,

$$\boldsymbol{\eta}^0 := \boldsymbol{\eta} + \boldsymbol{a} + b\boldsymbol{e} \wedge \mathbf{id} - \eta_3\boldsymbol{d} + (\boldsymbol{d} \cdot \mathbf{id})\boldsymbol{e} - \frac{1}{2}(\boldsymbol{d} \cdot \mathbf{id})\boldsymbol{d} \tag{14}$$

is also a solution to Eqs. (3)–(4). Let $\boldsymbol{d} := (-(\int_\omega \mathrm{d}\omega)^{-1}\int_\omega \partial_\alpha \eta_3\,\mathrm{d}\omega)$, so that $\int_\omega \partial_\alpha \eta_3^0\,\mathrm{d}\omega = 0$; it is then easily seen that there exist $\boldsymbol{a} \in \mathbb{R}^3$ and $b \in \mathbb{R}$ such that the corresponding vector field $\boldsymbol{\eta}^0$ (as defined in (14)) belongs to the space $\boldsymbol{V}^0(\omega)$.

To show that such a vector field $\boldsymbol{\eta}^0$ is unique, assume that $\tilde{\boldsymbol{\eta}}^0 \in \boldsymbol{V}^0(\omega)$ also satisfies Eqs. (3)–(4), so that $\tilde{\boldsymbol{\eta}}^0$ is necessarily of the form

$$\tilde{\boldsymbol{\eta}}^0 = \boldsymbol{\eta}^0 + \boldsymbol{a} + b\boldsymbol{e} \wedge \mathbf{id} - \eta_3\boldsymbol{d} + (\boldsymbol{d} \cdot \mathbf{id})\boldsymbol{e} - \frac{1}{2}(\boldsymbol{d} \cdot \mathbf{id})\boldsymbol{d}$$

for some $\boldsymbol{a} = (\boldsymbol{a}_H, a_3) \in \mathbb{R}^3$, $b \in \mathbb{R}$, and $\boldsymbol{d} \in \mathbb{R}^2$. It is then easily seen, first that $\boldsymbol{d} = \mathbf{0}$, then that $a_3 = 0$, $b = 0$, and $\boldsymbol{a}_H = \mathbf{0}$. □

3. Continuity of the mapping $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega) \to \boldsymbol{\eta} \in \boldsymbol{V}^0(\omega)$

We have the following nonlinear analog of Theorem 4.1 of [1]. The spaces $\mathbb{E}(\omega)$ and $\boldsymbol{V}^0(\omega)$ are those defined in (12) and (13).

Theorem 3.1. *Let ω be a simply-connected domain, and let*

$$\boldsymbol{\Phi} : \mathbb{E}(\omega) \to \boldsymbol{V}^0(\omega)$$

be the nonlinear bijection defined for each $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega)$ by $\boldsymbol{\Phi}(\boldsymbol{F}, \boldsymbol{F}) := \boldsymbol{\eta}$, where $\boldsymbol{\eta}$ is the unique element in the space $\boldsymbol{V}^0(\omega)$ that satisfies Eqs. (3)–(4) (*Theorem* 2.2). *Then there exists a constant C such that*

$$\left\| \boldsymbol{\Phi}(\boldsymbol{E}, \boldsymbol{F}) \right\|_{H^1(\omega) \times H^1(\omega) \times H^2(\omega)} \leqslant C \left(\|\boldsymbol{E}\|_{\mathbb{L}^2(\omega)} + \|\boldsymbol{F}\|_{\mathbb{L}^2(\omega)} + \|\boldsymbol{F}\|^2_{\mathbb{L}^2(\omega)} \right) \quad \text{for all } (\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega). \tag{15}$$

Besides, the set $\mathbb{E}(\omega)$ is sequentially weakly closed in $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$, and $\boldsymbol{\Phi}$ maps weakly convergent sequences in $\mathbb{E}(\omega)$ endowed with the topology of $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ into strongly convergent sequences in $\boldsymbol{V}^0(\omega)$ endowed with the topology of $L^2(\omega) \times L^2(\omega) \times H^1(\omega)$.

Sketch of proof. That the nonlinear mapping $\boldsymbol{\Phi}$ is a bijection from $\mathbb{E}(\omega)$ onto $\boldsymbol{V}^0(\omega)$ follows from necessity of the nonlinear Saint-Venant compatibility conditions, and from their sufficiency (established in Theorem 2.2). Besides, for each $\boldsymbol{\eta} = (\boldsymbol{\eta}_H, \eta_3) \in \boldsymbol{V}^0(\omega)$,

$$\boldsymbol{\Phi}^{-1}(\boldsymbol{\eta}) = \left(\nabla_s \boldsymbol{\eta}_H + \frac{1}{2} \nabla \eta_3 \nabla \eta_3^T, \nabla^2 \eta_3 \right).$$

Given any $\boldsymbol{\eta} = (\boldsymbol{\eta}_H, \eta_3) \in \boldsymbol{V}^0(\omega)$, the function $\eta_3 \in H^2(\omega)$ satisfies $\int_\omega \eta_3 \, d\omega = \int_\omega \partial_\alpha \eta_3 \, d\omega = 0$. Hence the Poincaré–Wirtinger inequality implies that there exists a constant C_1 such that

$$\|\eta_3\|_{H^2(\omega)} \leqslant C_1 \left\| \nabla^2 \eta_3 \right\|_{\mathbb{L}^2(\omega)} \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}^0(\omega). \tag{16}$$

Writing $\nabla_s \boldsymbol{\eta}_H = (\nabla_s \boldsymbol{\eta}_H + \frac{1}{2} \nabla \eta_3 \nabla \eta_3^T) - \frac{1}{2} \nabla \eta_3 (\nabla \eta_3)^T$, we then infer from the classical two-dimensional Korn's inequality that there exists a constant C_2 such that

$$\|\boldsymbol{\eta}_H\|_{H^1(\omega) \times H^1(\omega)} \leqslant C_2 \left\| \nabla_s \boldsymbol{\eta}_H + \frac{1}{2} \nabla \eta_3 \nabla \eta_3^T \right\|_{\mathbb{L}^2(\omega)} + \left\| \nabla \eta_3 (\nabla \eta_3)^T \right\|_{\mathbb{L}^2(\omega)} \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}^0(\omega). \tag{17}$$

Given any $\boldsymbol{\eta} = (\boldsymbol{\eta}_H, \eta_3) \in \boldsymbol{V}^0(\omega)$, the vector field $\nabla \eta_3 \in H^1(\omega) \times H^1(\omega)$ satisfies $\int_\omega \nabla \eta_3 \, d\omega = \mathbf{0}$; besides, the continuous injection $H^1(\omega) \hookrightarrow L^4(\omega)$ holds. Hence there exist constants C_3 and C_4 such that

$$\|\nabla \eta_3\|_{\mathbf{L}^4(\omega)} \leqslant C_3 \|\nabla \eta_3\|_{\boldsymbol{H}^1(\omega)} \leqslant C_4 \|\eta_3\|_{H^2(\omega)} \leqslant C_1 C_4 \left\| \nabla^2 \eta_3 \right\|_{\mathbb{L}^2(\omega)} \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}^0(\omega). \tag{18}$$

Since, finally, there exists a constant C_5 such that

$$\left\| \nabla \eta_3 (\nabla \eta_3)^T \right\|_{\mathbb{L}^2(\omega)} \leqslant C_5 \left(\|\nabla \eta_3\|_{\mathbf{L}^4(\omega)} \right)^2 \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{V}^0(\omega), \tag{19}$$

inequality (15) follows by combining the above inequalities.

In what follows, $\to$, resp. $\rightharpoonup$, denotes strong, resp. weak, convergence. Let $(\boldsymbol{E}^k, \boldsymbol{F}^k) \in \mathbb{E}(\omega)$, $k \geqslant 1$, and $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ be such that

$$\left(\boldsymbol{E}^k, \boldsymbol{F}^k \right) \rightharpoonup (\boldsymbol{E}, \boldsymbol{F}) \quad \text{in } \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega) \text{ as } k \to \infty.$$

By inequality (15), the sequence $(\boldsymbol{\eta}^k)_{k=1}^\infty$, where $\boldsymbol{\eta}^k := \boldsymbol{\Phi}(\boldsymbol{E}^k, \boldsymbol{F}^k) \in \boldsymbol{V}^0(\omega)$ is then bounded in $\boldsymbol{V}^0(\omega)$. Since $\boldsymbol{V}^0(\omega)$ is reflexive (as a closed subspace of $H^1(\omega) \times H^1(\omega) \times H^2(\omega)$), there exists a subsequence $(\boldsymbol{\eta}^\ell)_{\ell=1}^\infty$ and $\boldsymbol{\eta} \in \boldsymbol{V}^0(\omega)$ such that

$$\boldsymbol{\eta}^\ell \rightharpoonup \boldsymbol{\eta} \quad \text{in } H^1(\omega) \times H^1(\omega) \times H^2(\omega) \quad \text{and} \quad \boldsymbol{\eta}^\ell \to \boldsymbol{\eta} \quad \text{in } L^2(\omega) \times L^2(\omega) \times H^1(\omega).$$

Hence $F^k_{\alpha\beta} = \partial_{\alpha\beta} \eta^k_3 \rightharpoonup \partial_{\alpha\beta} \eta_3$ in $L^2(\omega)$, which shows that $F_{\alpha\beta} = \partial_{\alpha\beta} \eta_3$ (uniqueness of the weak limit). Furthermore $\eta^\ell_3 \to \eta_3$ in $H^1(\omega)$ implies $\partial_\alpha \eta^\ell_3 \to \partial_\alpha \eta_3$ in $L^2(\omega)$, so that $\partial_\alpha \eta^\ell_3 \partial_\beta \eta^\ell_3 \to \partial_\alpha \eta_3 \partial_\beta \eta_3$ in $L^1(\omega)$. Since $\frac{1}{2}(\partial_\alpha \eta^\ell_\beta + \partial_\beta \eta^\ell_\alpha) \rightharpoonup \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha)$ in $L^2(\omega)$, it follows that, for each $\varphi \in \mathcal{D}(\omega)$,

$$\int_\omega E^k_{\alpha\beta} \varphi \, d\omega \to \int_\omega \frac{1}{2} (\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\eta \eta_3) \varphi \, d\omega,$$

which shows that $E_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \alpha_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3)$. Consequently, $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega)$ since $\boldsymbol{\eta} \in \boldsymbol{V}^0(\omega)$. Therefore $\mathbb{E}(\omega)$ is sequentially weakly closed.

Finally, the uniqueness of the limit implies that the whole sequence $\boldsymbol{\eta}^k$ strongly converges to $\boldsymbol{\eta}$ in $L^2(\omega) \times L^2(\omega) \times H^1(\omega)$. $\square$

Note that, when equivalently expressed in terms of the vector fields $\boldsymbol{\eta} \in \boldsymbol{V}^0(\omega)$ (instead of the matrix fields $(\boldsymbol{E}, \boldsymbol{F})$ in the space $\mathbb{E}(\omega)$ of (12), inequality (15) provides an instance of a *nonlinear Korn's inequality*.

In [3], Theorem 3.1 will be put to use for establishing the existence of a minimizer over the space $\mathbb{E}(\omega)$ of the functional $\mathcal{J} : \mathbb{E}(\omega) \to \mathbb{R}$ defined for each $(\boldsymbol{E}, \boldsymbol{F}) \in \mathbb{E}(\omega)$ by

$$\mathcal{J}(\boldsymbol{E}, \boldsymbol{F}) := \frac{1}{2} \int_\omega \left\{ \varepsilon a_{\alpha\beta\sigma\tau} E_{\sigma\tau} E_{\alpha\beta} + \frac{\varepsilon^3}{3} a_{\alpha\beta\sigma\tau} F_{\sigma\tau} F_{\alpha\beta} \right\} \mathrm{d}\omega - L\big(\boldsymbol{\Phi}(\boldsymbol{E}, \boldsymbol{F})\big), \tag{20}$$

when $\boldsymbol{p}_H = \boldsymbol{0}$ (if $\boldsymbol{p}_H \neq \boldsymbol{0}$, a vector field in $\mathbb{R}^2$ must be introduced as an extra variable; cf. [3]), thereby justifying the *intrinsic approach* for the *Neumann problem* described in Section 1. Besides, the convexity of the integrand in the functional $\mathcal{J}$ of (20) with respect to its arguments $\boldsymbol{E} = (E_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ and $\boldsymbol{F} = (F_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ will lay the ground for defining a notion of *polyconvexity* adapted to the Kirchhoff–von Kármán–Love theory of nonlinearly elastic plates.

Acknowledgement

The work described in this paper was partially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region [Project No. City U 100709].

References

[1] P.G. Ciarlet, P. Ciarlet Jr., Another approach to linearized elasticity and a new proof of Korn's inequality, Math. Models Methods Appl. Sci. 15 (2005) 259–271.
[2] P.G. Ciarlet, P. Destuynder, A justification of a nonlinear model in plate theory, Computer Methods Appl. Mech. Engrg. 17/18 (1979) 227–258.
[3] P.G. Ciarlet, S. Mardare, An intrinsic approach and a notion of polyconvexity for nonlinearly elastic plates, C. R. Acad. Sci. Paris, Ser. I, doi:10.1016/j.crma.2011.11.001, in press.
[4] P.G. Ciarlet, S. Mardare, Saint-Venant compatibility conditions and a notion of polyconvexity in nonlinear plate theory, in preparation.
[5] G. Friesecke, R.D. James, S. Müller, A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence, Arch. Rational Mech. Anal. 180 (2006) 183–236.
[6] T. von Kármán, Festigkeitsprobleme im Maschinenbau, in: Encyclopädie der Mathematischen Wissenschaften, vol. IV/4, Leipzig, 1910, pp. 311–385.
[7] G. Kirchhoff, Über das Gleichgewicht und die Bewegung einer elastischen Scheibe, J. Reine Angew. Math. 40 (1850) 51–58.
[8] A.E.H. Love, Treatise on the Mathematical Theory of Elasticity, fourth edition, Cambridge University Press, Cambridge, 1934 (reprinted by Dover Publications, New York, 1944).
[9] J. Nečas, J. Naumann, On a boundary value problem in nonlinear theory of thin elastic plates, Aplikace Matematiky 19 (1974) 7–16.
[10] P. Rabier, Résultats d'existence dans des modèles non linéaires de plaques, C. R. Acad. Sci. Paris, Sér. A 289 (1979) 515–518.