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Testing instantaneous linear Granger causality in presence of nonlinear dynamics

Test de la causalité instantanée linéaire de Granger en présence de dynamiques non linéaires

Hamdi Raïssi

20. avenue des buttes de Coësmes. CS 70839. 35708 Rennes cedex 7. France

ARTICLE INFO Article history: This Note is devoted to the test of instantaneous linear Granger causality when the errors Received 11 July 2011 are dependent but uncorrelated. The assumptions are weak and include a large set of Accepted after revision 10 October 2011 dynamics as for instance the GARCH processes. We show that the standard Wald test Available online 21 October 2011 for testing instantaneous linear Granger causality is not valid in our framework. As a

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ABSTRACT

consequence Wald tests which are valid in our framework are proposed.

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RÉSUMÉ

Dans cette Note on considère le test de la causalité instantanée linéaire au sens de Granger entre deux variables dans le cas où les innovations sont dépendantes mais non corrélées (c'est-à-dire des innovations linéaires). Les hypothèses considérées sont faibles et peuvent prendre en compte des non linéarités comme par exemple celles induites par les processus GARCH. Nous établissons que la statistique de Wald standard pour tester la causalité instantanée linéaire n'est pas valide dans notre cadre. En conséquence des tests de Wald valides sont proposés.

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1. Introduction

Nonlinear processes are commonly considered in the literature. For instance processes which exhibit nonlinear dynamics and admit a linear representation with dependent errors are presented in [6], or [1]. Therefore asserting that all the dynamics of a time series are captured by a linear model is restrictive in many cases, so that instantaneous linear causality is often tested in presence of nonlinear dynamics of unknown form. Weak VAR models (i.e. VAR models with dependent but uncorrelated errors) were studied by [5]. They established the asymptotic normality of the Ordinary Least Squares (OLS) estimators. Using this result we propose in this Note modified Wald tests which take into account the important case of VAR models with dependent but uncorrelated errors. It is found that the test with modified statistic achieves a gain in power, in the Bahadur sense, when compared to the test with modified asymptotic distribution.

E-mail address: hamdi.raissi@insa-rennes.fr.

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2. Asymptotic results

Let us consider the VAR model of order $p \ge 0$ and dimension d

$$X_t = A_{01}X_{t-1} + \dots + A_{0p}X_{t-p} + u_t, \tag{1}$$

where the matrices A_i are such that det $A(z) \neq 0$ for all $|z| \leq 1$ with $A(z) = I_d - \sum_{i=1}^p A_i z^i$ and I_d denotes the $d \times d$ identity matrix. We assume that X_1, \ldots, X_T are observed. The errors are usually assumed iid (Gaussian). However it is well known in the literature that there are numerous situations where the errors are martingale differences (as for the GARCH models) or where the optimal predictor of the observed process does not even correspond to the best linear predictor (as when the innovations are driven by an All-Pass model). Therefore we only assume that the error process is uncorrelated and control its dependence. Define the mixing coefficients

$$\alpha_a(h) = \sup_{A \in \sigma(a_u, u \leq t), B \in \sigma(a_u, u \geq t+h)} \left| P(A \cap B) - P(A)P(B) \right|$$

that measure the temporal dependence of the stationary process (a_t) , and introduce $||a_t||_q = (E||a_t||^q)^{1/q}$, where ||.|| denotes the Euclidean norm with $E||a_t||^q < \infty$.

Assumption A1. (i) The process (u_t) is strictly stationary ergodic with positive definite covariance matrix Ω_u and such that $Eu_t = 0$, $Cov(u_t, u_{t-h}) = 0$ for all $t \in \mathbb{Z}$ and all $h \neq 0$.

(ii) The process (u_t) satisfies $||u_t||_{4+2\nu} < \infty$, and the mixing coefficients of the process (u_t) are such that $\sum_{h=0}^{\infty} \{\alpha_u(h)\}^{\nu/(2+\nu)} < \infty$ for some $\nu > 0$.

Note that the framework given by Assumption A1 is quite general and includes multivariate GARCH models. In particular the mixing assumption in (ii) is not very restrictive and is valid for a large class of processes (see *e.g.* [4]).

The parameters of model (1) can be estimated by OLS. Let us introduce the vector of the autoregressive parameters $\theta_0 = \text{vec}(A_{01}, \ldots, A_{0p}), \tilde{X}_{t-1} = (X'_{t-1}, \ldots, X'_{t-p})'$ and the OLS estimators $\hat{\theta}$

$$\hat{\theta} = \operatorname{vec}(\hat{\Sigma}_{X_t, \tilde{X}_{t-1}} \hat{\Sigma}_{\tilde{X}_{t-1}}^{-1}),$$

where

$$\hat{\Sigma}_{X_t,\tilde{X}_{t-1}} = T^{-1} \sum_{t=p+1}^T X_t \tilde{X}_{t-1}', \qquad \hat{\Sigma}_{\tilde{X}_{t-1}} = T^{-1} \sum_{t=p+1}^T \tilde{X}_{t-1} \tilde{X}_{t-1}'$$

and vec(.) correspond to the column vectorization operator of a matrix. We also define $\hat{\Omega}_u = T^{-1} \sum_{t=1}^T \hat{u}_t \hat{u}'_t$, where $\hat{u}_t = u_t(\hat{\theta}) = X_t - (\tilde{X}'_{t-1} \otimes I_d)\hat{\theta}$ are the residuals. It is shown in [5] that $\sqrt{T}(\hat{\theta} - \theta_0) = O_p(1)$ and $\hat{\Omega}_u - \Omega_u = o_p(1)$ under Assumption A1.

Introduce $Y_t = (X_{1,t}, \ldots, X_{d_1,t})'$ of dimension d_1 and $Z_t = (X_{d_1+1,t}, \ldots, X_{d,t})'$ of dimension $d_2 = d - d_1$. We follow the notations of [8]. Denote by $Y_t(X_s | s \leq t)$ the optimal one step linear predictor of Y_{t+1} at the date t, based on the information of the present and the past of the process (X_t) . Denote also by $\Sigma_y(X_s | s \leq t)$ the corresponding forecast mean squared error. Similarly we define $Y_t(X_s \cup \{Z_{t+1}\} | s \leq t)$ and $\Sigma_y(X_s \cup \{Z_{t+1}\} | s \leq t)$. It is said that there is instantaneous linear causality between (Y_t) and (Z_t) if

$$\Sigma_{\mathcal{Y}}(X_s \cup \{Z_{t+1}\} | s \leqslant t) \neq \Sigma_{\mathcal{Y}}(X_s | s \leqslant t).$$

It is well known that there is no instantaneous linear causality between (Y_t) and (Z_t) if and only if the components Ω_u^{ij} of Ω_u with $j \leq d_1$ and $i > d_1$ are equal to zero. Therefore testing for instantaneous linear causality between (Y_t) and (Z_t) amounts to test the following pair of hypotheses:

$$H_0: R \operatorname{vech}(\Omega_u) = 0$$
 vs $R \operatorname{vech}(\Omega_u) \neq 0$,

where the vech(.) operator stacks the elements on and below the main diagonal of a square matrix and R is a suitable restriction matrix. The test of instantaneous linear causality between the processes (Y_t) and (Z_t) in the non-standard framework of Assumption A1 is studied.

In this part we introduce the tests for instantaneous linear causality. Define

$$\Psi_s = \{\operatorname{vech}(E(u_tu'_t))\operatorname{vech}(E(u_tu'_t))'\} \text{ and } \Psi_m = \sum_{h=-\infty}^{\infty} E\{\operatorname{vech}(u_tu'_t)\operatorname{vech}(u_{t-h}u'_{t-h})'\}.$$

The matrix Ψ_s is consistently estimated by $\hat{\Psi}_s = \{\operatorname{vech}(\hat{\Omega}_u) \operatorname{vech}(\hat{\Omega}_u)'\}$ while the matrix Ψ_m can be consistently estimated using the HAC estimation method (see [6] and references therein). Let us denote by $\hat{\Psi}_m$ such consistent estimator. In the sequel it is assumed that $R\Psi_m R'$ is invertible, so that $R\hat{\Psi}_m R'$ is invertible at least asymptotically.

Now consider the following statistic which is asymptotically distributed as a $\chi^2_{d_1d_2}$ under the assumption of iid errors

$$S_{s} = T \delta_{T}^{\prime} \left(R \hat{\Psi}_{s} R^{\prime} \right)^{-1} \delta_{T},$$

where $\delta_T = R \operatorname{vech}(\hat{\Omega}_u)$. Then using the statistic S_s one can define a test for testing instantaneous linear causality with standard critical values. This standard Wald test will be denoted by W_s . We introduce the modified statistic

$$S_m = T \delta_T' \left(R \hat{\Psi}_m R' \right)^{-1} \delta_T.$$

The following proposition gives the asymptotic distribution of S_m and highlights that the W_s test is in general unreliable when the innovations are dependent. Weak convergence is denoted by $\stackrel{d}{\rightarrow}$.

Proposition 1. Under Assumption A1 we have

$$S_m \xrightarrow{d} \chi^2_{d_1 d_2}$$
 (2)

and

$$S_s \stackrel{d}{\to} \sum_{i=1}^{d_1 d_2} \lambda_i U_i^2, \tag{3}$$

as $T \to \infty$, where the λ_i 's correspond to the eigenvalues of the matrix

$$\Delta = \left(R\Psi_{s}R'\right)^{-1/2}\left(R\Psi_{m}R'\right)\left(R\Psi_{s}R'\right)^{-1/2}$$

and the U_i 's are independent $\mathcal{N}(0, 1)$ variables.

Using the result (2) we can define a modified test based on standard critical values which will be denoted as W_m . If we assume that the error process is iid we have $\Psi_m = \Psi_s$ and $\Delta = I_{d_1d_2}$, so that we retrieve that $S_s \stackrel{d}{\rightarrow} \chi^2_{d_1d_2}$. However this is not the case in general and using the standard Wald test can be quite misleading when the errors are dependent. It is clear that using the $\hat{\Psi}_m$ and $\hat{\Psi}_s$ estimators one can define consistent estimators $\hat{\lambda}_i$ of the eigenvalues λ_i . Then using the result (3) we introduce a test \widehat{W}_m based on the statistic S_s which is valid in the framework of Assumption A1. At the asymptotic level α , this test consists in rejecting the null hypothesis of no instantaneous linear causality between (Y_t) and (Z_t) if the *p*-value is such that

$$P\left(S_s < \sum_{i=1}^{d_1d_2} \hat{\lambda}_i U_i^2 \mid X_1, \ldots, X_T\right) < \alpha.$$

Therefore the *p*-values are computed directly from the data and can be evaluated using the Imhof algorithm [7].

In this part we investigate the power properties of the studied tests. To this aim we use the approximate Bahadur slope approach [3]. For the tests based on the S_s statistic define $q_s(x) = P(\sum_{i=1}^{d_1d_2} \lambda_i U_i^2 > x)$ for any x > 0. For a fixed alternative R vech $\Omega_u = \omega \neq 0$ consider the asymptotic slope $c_s(\omega) = 2 \lim_{T \to \infty} T^{-1}q_s(S_s)$. Define similarly $c_m(\omega)$ for the W_m and also the asymptotic relative efficiency $ARE_{S_m,S_s}(\omega) = c_m(\omega)/c_s(\omega)$. A relative efficiency $ARE_{S_m,S_s}(\omega) \ge 1$ suggests that the W_m test is more able to detect the alternative ω than the tests based on the S_s statistic. In such a case the p-values of the W_m test converge faster to zero than those of the W_s and \widetilde{W}_m tests.

Proposition 2. Under Assumption A1 we have $ARE_{S_m, S_s}(\omega) \ge 1$ for every $\omega \in \mathbb{R}^{d_1d_2} \setminus \{0\}$.

Note that if we suppose that the errors are iid we obtain $ARE_{S_m,S_s}(\omega) = 1$ for any alternative $\omega \neq 0$ and hence there is no loss of power in the standard case for the W_m when compared to the W_s test in the Bahadur sense. On the other hand the W_m has a power advantage on the W_s in the non-standard case. In addition the W_m which has standard asymptotic distribution and achieves a gain in power must be preferred to the \widetilde{W}_m test. However it is well known that inverting HAC matrices in statistics can entail size distortions as pointed out by [2]. Therefore we can expect that the gain in power for the W_m test comes at the cost of a bad control of type I errors.

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