C. R. Acad. Sci. Paris, Ser. I 349 (2011) 1251–1254

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial Differential Equations

A canonical extension of Korn's first inequality to H(Curl) motivated by gradient plasticity with plastic spin

Une extension canonique de l'inégalité de Korn à H(Curl) *motivée par un modèle de plasticité à gradient avec rotation plastique*

Patrizio Neff, Dirk Pauly, Karl-Josef Witsch

Universität Duisburg-Essen, Fakultät für Mathematik, Campus Essen Universitätsstr. 2, 45117 Essen, Germany

ARTICLE INFO

Article history:
Received 5 July 2011
Accepted 22 September 2011
Available online 23 November 2011

Presented by Philippe G. Ciarlet

ABSTRACT

We prove a Korn-type inequality in $\overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$ for tensor fields P mapping Ω to $\mathbb{R}^{3\times 3}$. More precisely, let $\Omega\subset\mathbb{R}^3$ be a bounded domain with connected Lipschitz boundary $\partial\Omega$. Then, there exists a constant $c>0$ such that

$$c\|P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}\leqslant\|\operatorname{sym}P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}+\|\operatorname{Curl}P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}\tag{1}$$

holds for all tensor fields $P\in\overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$, i.e., all $P\in\mathsf{H}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$ with vanishing tangential trace on $\partial\Omega$. Here, rotation and tangential traces are defined row-wise. For compatible P, i.e., $P=\nabla v$ and thus $\operatorname{Curl}P=0$, where $v\in\mathsf{H}^1(\Omega,\mathbb{R}^3)$ are vector fields having components v_n, for which ∇v_n are normal at $\partial\Omega$, the presented estimate (1) reduces to a non-standard variant of Korn's first inequality, i.e.,

$$c\|\nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}\leqslant\|\operatorname{sym}\nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}.$$

On the other hand, for skew-symmetric P, i.e., $\operatorname{sym}P=0$, (1) reduces to a non-standard version of Poincaré's estimate. Therefore, since (1) admits the classical boundary conditions our result is a common generalization of these two classical estimates, namely Poincaré's resp. Korn's first inequality.

RÉSUMÉ

Nous démontrons une inégalité de type Korn dans $\overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$ pour des champs tensoriels P appliquant Ω dans $\mathbb{R}^{3\times 3}$. De façon plus précise, soit Ω un domaine borné de $\mathbb{R}^3$ dont la frontière $\partial\Omega$ est Lipschitz continue et connexe. Il existe alors une constante $c>0$, telle que

$$c\|P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}\leqslant\|\operatorname{sym}P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}+\|\operatorname{Curl}P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}\tag{1}$$

est vérifiée pour tous les champs tensoriels $P\in\overset{\circ}{\mathsf{H}}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$, i.e., pour tous les $P\in\mathsf{H}(\operatorname{Curl};\Omega,\mathbb{R}^{3\times 3})$ dont la trace tangentielle s'annule sur $\partial\Omega$. Ici, rotation et trace tangentielle sont définies ligne par ligne. Pour des champs P compatibles, i.e., $P=\nabla v$, d'où $\operatorname{Curl}P=0$, avec $v\in\mathsf{H}^1(\Omega,\mathbb{R}^3)$ et de composante v_n, telle que ∇v_n est normal à $\partial\Omega$,

E-mail addresses: patrizio.neff@uni-due.de (P. Neff), dirk.pauly@uni-due.de (D. Pauly), kj.witsch@uni-due.de (K.-J. Witsch).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.10.003

l'estimation (1) se réduit à

$$c\|\nabla v\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{3\times 3})} \leqslant \|\operatorname{sym}\nabla v\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{3\times 3})},$$

une variante non classique de la première inégalité de Korn. Par ailleurs, pour des P anti-symétriques, (1) se réduit à une variante non classique de l'inégalité de Poincaré. Il en résulte que puisque (1) est compatible avec les conditions aux limites classiques, cette estimation généralise tout à la fois l'inégalité de Poincaré et la première inégalité de Korn.

1. Introduction

The motivation for our new estimate is a formulation of infinitesimal gradient plasticity [2]. Our model is taken from Neff et al. [6]. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. The goal is to find the displacement $u : [0,\infty) \times \Omega \mapsto \mathbb{R}^3$ and the possibly non-symmetric plastic distortion tensor $P : [0,\infty) \times \Omega \mapsto \mathbb{R}^{3\times 3}$, such that in $[0,\infty) \times \Omega$

$$\operatorname{Div}\sigma = f, \quad \sigma = 2\mu\operatorname{sym}(\nabla u - P) + \lambda\operatorname{tr}(\nabla u - P)\,\mathrm{id}, \tag{2}$$

$$\dot{P} \in \Phi(\Sigma), \quad \Sigma = \sigma - 2\mu\operatorname{sym}P - \mu L_c^2\operatorname{Curl}\operatorname{Curl}P, \tag{3}$$

hold. The system is completed by the boundary conditions

$$u(t,x) = 0, \qquad \nu(x) \times P(t,x) = 0 \quad \forall (t,x) \in [0,\infty) \times \partial\Omega \tag{4}$$

and the initial condition $P(0,x) = 0$ for all $x \in \Omega$. The underlying thermodynamic potential including the plastic gradients in form of the dislocation density tensor $\operatorname{Curl}P$ is

$$\int_\Omega \mu\left|\operatorname{sym}(\nabla u - P)\right|^2 + \frac{\lambda}{2}\left|\operatorname{tr}(\nabla u - P)\right|^2 - f\cdot u + \mu|\operatorname{sym}P|^2 + \frac{\mu}{2}L_c^2|\operatorname{Curl}P|^2.$$

Here, μ, λ are the elastic Lamé moduli and σ is the symmetric Cauchy stress tensor. The system is driven by nonzero body forces denoted by f. The exterior normal to the boundary $\partial\Omega$ is denoted by ν and the plastic distortion P is required to satisfy row-wise the homogeneous tangential boundary condition which means that the boundary $\partial\Omega$ is a perfect conductor regarding the plastic distortion.[1]

Moreover, $\Phi : \mathbb{R}^{3\times 3} \mapsto \mathbb{R}^{3\times 3}$ is the monotone, multivalued flow-function with $\Phi(0) = 0$ and $\Phi(\mathbb{R}^{3\times 3}_{\mathrm{sym}}) \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. In general, Σ is not symmetric even if P is symmetric. Thus, the plastic inhomogeneity is responsible for the plastic spin (the possible non-symmetry of P). The mathematically suitable space for symmetric plastic distortion P is the classical space $\mathrm{H}(\mathrm{curl};\Omega)$ for each row of P [10,2]. This case appears when choosing $\Phi : \mathbb{R}^{3\times 3} \mapsto \mathbb{R}^{3\times 3}_{\mathrm{sym}}$.

In the large scale limit $L_c \to 0$ we recover a classical elasto-plasticity model with local kinematic hardening and symmetric plastic strain $\varepsilon_p := \operatorname{sym}P$, since then $\dot{P} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$.

Uniqueness of classical solutions for rate-independent and rate-dependent formulations of this model is shown in [6]. The more difficult existence question for the rate-independent model in terms of a weak reformulation is addressed in [6]. First numerical results for a simplified rate-independent irrotational formulation (no plastic spin, i.e., symmetric plastic distortion P) are presented in [10], cf. [15]. In [3] the model has been extended to rate-independent isotropic hardening based on the concept of a dissipation function defined in terms of the equivalent plastic strain. From a modeling point of view, it is strongly preferable to again have only the symmetric (rate) part of the plastic distortion P appear in the dissipation potential.

The existence and uniqueness can be settled by recasting the model as a variational inequality, if it is possible to define a bilinear form which is coercive with respect to appropriate spaces. This program has been achieved for other variants of the model in [3]. It had to remain basically open for the above system (2)–(4). In this case, the appropriate space for the plastic distortion P is the completion $\overset{\circ}{\mathrm{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$ of the linear space

$$\left\{P \in \mathrm{C}^\infty\left(\overline{\Omega},\mathbb{R}^{3\times 3}\right):\ P_n \text{ normal at } \partial\Omega,\ n = 1,2,3\right\}$$

with respect to the norm $|\!|\!|\cdot|\!|\!|$, where P_n are the columns of P^T and

$$|\!|\!|P|\!|\!|^2 := \|\operatorname{sym}P\|^2_{\mathrm{L}^2(\Omega)} + \|\operatorname{Curl}P\|^2_{\mathrm{L}^2(\Omega)}.$$

Despite first appearance, this quadratic form indeed defines a norm as shown in [6]. Thus $\overset{\circ}{\mathrm{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega)$ is a Hilbert-space. However, in this space it is not immediately obvious how to define a linear and bounded tangential trace operator. Since only $\|\operatorname{sym}P\|_{\mathrm{L}^2(\Omega)}$ appears, it is also not clear, how to control the skew-symmetric part of P. Therefore, the crucial embedding

[1] This homogeneous tangential boundary condition on P is consistent with $\nu \times \nabla u = 0$ on $\partial\Omega$ which follows from $u = 0$ on $\partial\Omega$.

$$\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) \subset \mathsf{L}^2(\Omega)$$

is not clear as well. As a consequence of our main results of this note, i.e., Theorems 1 and 2, we obtain that nevertheless and fortunately

$$\overset{\circ}{\mathsf{H}}_{\mathrm{sym}}(\mathrm{Curl};\Omega) = \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$$

holds with equivalent norms in case the domain Ω has a connected Lipschitz boundary.

The result of this note has been announced in [7] and is written down more detailed in [9]. A forthcoming paper [8] will be devoted to the case $\Omega \subset \mathbb{R}^N$ using differential forms.

For the proof of our main result (1), i.e., Theorems 1 and 2, we combine techniques from electro-magnetic and elastic theory, namely the Helmholtz decomposition, the Maxwell or Poincaré/Friedrichs' estimate and Korn's first inequality. Their basic variants are well known results which can be found in many books, e.g., in [4] and the literature cited there. More sophisticated and related versions are presented, e.g., in [11–14,16] for Maxwell's equations and [1,5] for Korn's inequality.

2. Results

Let Ω be a bounded domain in $\mathbb{R}^3$ with connected Lipschitz continuous boundary. We will denote the standard Sobolev spaces by $\mathsf{H}(\mathrm{grad};\Omega)$, $\mathsf{H}(\mathrm{div};\Omega)$, $\mathsf{H}(\mathrm{curl};\Omega)$ and introduce the differential operators Grad, Div, Curl as well as the corresponding vector resp. tensor (matrix) field Sobolev spaces

$$\mathsf{H}(\mathrm{Grad};\Omega), \quad \mathsf{H}(\mathrm{Div};\Omega), \quad \mathsf{H}(\mathrm{Curl};\Omega)$$

canonically by row-wise operation of the usual differential operators grad, div, curl. Equipped with their natural graph norms, these are Hilbert spaces. Furthermore, we define their closed subspaces

$$\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega), \quad \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$$

as completion (under the respective norms) of the vector resp. tensor valued space $\overset{\circ}{\mathsf{C}}{}^{\infty}(\Omega)$. An index 0 at the lower right corner indicates the vanishing of the differential operator, i.e.,

$$\mathsf{H}(\mathrm{Div}_0;\Omega) := \bigl\{T \in \mathsf{H}(\mathrm{Div};\Omega):\ \mathrm{Div}\,T = 0\bigr\}.$$

For tensor fields $T \in \mathsf{H}(\mathrm{Curl};\Omega)$ we define the semi-norm $|||\cdot|||$ by

$$|||T|||^2 := \|\operatorname{sym} T\|^2_{\mathsf{L}^2(\Omega)} + \|\operatorname{Curl} T\|^2_{\mathsf{L}^2(\Omega)}.$$

Theorem 1. *There exists a constant $c > 0$, such that for all $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$*

$$\|T\|_{\mathsf{L}^2(\Omega)} \leqslant c|||T|||.$$

Proof. Let $T \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$. Applying row-wise the well-known (orthogonal) Helmholtz decomposition, we get

$$T = \mathrm{Grad}\,v + S \in \mathrm{Grad}\,\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \oplus \mathsf{H}(\mathrm{Div}_0;\Omega).$$

Then, $\mathrm{Curl}\,T = \mathrm{Curl}\,S$ and we observe $S \in \overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega) \cap \mathsf{H}(\mathrm{Div}_0;\Omega)$ since

$$\mathrm{Grad}\,\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega) \subset \overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega).$$

By Poincaré/Friedrichs' estimate, there exists a constant $c_{pf} > 0$ independent of S and T, such that

$$\|S\|_{\mathsf{L}^2(\Omega)} \leqslant c_{pf}\|\operatorname{Curl} T\|_{\mathsf{L}^2(\Omega)}. \tag{5}$$

Then, by Korn's first inequality we obtain easily

$$\|T\|^2_{\mathsf{L}^2(\Omega)} = \|\operatorname{Grad} v\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} \leqslant 2\|\operatorname{sym}\operatorname{Grad} v\|^2_{\mathsf{L}^2(\Omega)} + \|S\|^2_{\mathsf{L}^2(\Omega)} \leqslant 4\|\operatorname{sym} T\|^2_{\mathsf{L}^2(\Omega)} + 5\|S\|^2_{\mathsf{L}^2(\Omega)}$$

and (5) completes the proof. □

We note that $c \leqslant \max\{2, \sqrt{5}c_{pf}\}$. The immediate consequence is

Theorem 2. *On* $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ *the norms* $\|\cdot\|_{\mathsf{H}(\mathrm{Curl};\Omega)}$ *and* $|||\cdot|||$ *are equivalent. In particular,* $|||\cdot|||$ *is a norm on* $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ *and*

$$\exists c>0\ \forall T\in\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)\quad c\|T\|_{\mathsf{H}(\mathrm{Curl};\Omega)}\leqslant\|\operatorname{sym}T\|_{\mathsf{L}^2(\Omega)}+\|\operatorname{Curl}T\|_{\mathsf{L}^2(\Omega)}.$$

Finally we note that

Remark 3. The estimate in Theorem 1, i.e.,

$$\|T\|_{\mathsf{L}^2(\Omega)}\leqslant c\big(\|\operatorname{sym}T\|^2_{\mathsf{L}^2(\Omega)}+\|\operatorname{Curl}T\|^2_{\mathsf{L}^2(\Omega)}\big)^{1/2},$$

is a common formulation of Korn's first and Poincaré's inequality.

(i) If $\operatorname{Curl}T=0$, we obtain Korn's first inequality. This, i.e.,

$$\|T\|_{\mathsf{L}^2(\Omega)}\leqslant c\|\operatorname{sym}T\|_{\mathsf{L}^2(\Omega)},$$

holds e.g. for tensor fields $T\in\overset{\circ}{\mathsf{H}}(\mathrm{Curl}_0;\Omega)$ or $T=\operatorname{Grad}v$ with vector fields $v\in\overset{\circ}{\mathsf{H}}(\mathrm{Grad};\Omega)$.

(ii) If $\operatorname{sym}T=0$, Poincaré's inequality appears. For skew-symmetric tensor fields $T\in\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega)$ we have

$$\|T\|_{\mathsf{L}^2(\Omega)}\leqslant c\|\operatorname{Curl}T\|_{\mathsf{L}^2(\Omega)}\leqslant 2c\|\nabla T\|_{\mathsf{L}^2(\Omega)}.$$

References

[1] P.G. Ciarlet, On Korn's inequality, Chinese Ann. Math. 31B (5) (2010) 607–618.
[2] J.K. Djoko, F. Ebobisse, A.T. McBride, B.D. Reddy, A discontinuous Galerkin formulation for classical and gradient plasticity. Part 1: Formulation and analysis, Comput. Methods Appl. Mech. Engrg. 196 (37) (2007) 3881–3897.
[3] F. Ebobisse, P. Neff, Rate-independent infinitesimal gradient plasticity with isotropic hardening and plastic spin, Math. Mech. Solids 15 (2010) 691–703.
[4] R. Leis, Initial Boundary Value Problems in Mathematical Physics, Teubner, Stuttgart, 1986.
[5] P. Neff, On Korn's first inequality with nonconstant coefficients, Proc. Roy. Soc. Edinburgh A 132 (2002) 221–243.
[6] P. Neff, K. Chełmiński, H.D. Alber, Notes on strain gradient plasticity. Finite strain covariant modelling and global existence in the infinitesimal rate-independent case, Math. Mod. Meth. Appl. Sci. (M3AS) 19 (2) (2009) 1–40.
[7] P. Neff, D. Pauly, K.-J. Witsch, A Korn's inequality for incompatible tensor fields, in: Proceedings in Applied Mathematics and Mechanics (PAMM), 2011.
[8] P. Neff, D. Pauly, K.-J. Witsch, Maxwell meets Korn: A new coercive inequality for tensor fields in $\mathbb{R}^{N\times N}$ with square-integrable exterior derivative, Math. Methods Appl. Sci., submitted for publication, http://arxiv.org/abs/1105.5013; Preprint SE-E-737, Universität Duisburg-Essen, Schriftenreihe der Fakultät für Mathematik, 2011, http://www.uni-due.de/mathematik/preprints.shtml.
[9] P. Neff, D. Pauly, K.-J. Witsch, Poincaré meets Korn via Maxwell, J. Math. Sci. (N.Y.), submitted for publication, http://arxiv.org/abs/1106.4731; Preprint SE-E-736, Universität Duisburg-Essen, Schriftenreihe der Fakultät für Mathematik, 2011, http://www.uni-due.de/mathematik/preprints.shtml.
[10] P. Neff, A. Sydow, C. Wieners, Numerical approximation of incremental infinitesimal gradient plasticity, Internat. J. Numer. Methods Engrg. 77 (3) (2009) 414–436, http://www.mathematik.uni-karlsruhe.de/iwrmm/seite/preprints/media.
[11] D. Pauly, Hodge-Helmholtz decompositions of weighted Sobolev spaces in irregular exterior domains with inhomogeneous and anisotropic media, Math. Methods Appl. Sci. 31 (2008) 1509–1543.
[12] R. Picard, An elementary proof for a compact imbedding result in generalized electromagnetic theory, Math. Z. 187 (1984) 151–164.
[13] R. Picard, Some decomposition theorems and their applications to non-linear potential theory and Hodge theory, Math. Methods Appl. Sci. 12 (1990) 35–53.
[14] R. Picard, N. Weck, K.-J. Witsch, Time-harmonic Maxwell equations in the exterior of perfectly conducting, irregular obstacles, Analysis (Munich) 21 (2001) 231–263.
[15] B.D. Reddy, F. Ebobisse, A.T. McBride, Well-posedness of a model of strain gradient plasticity for plastically irrotational materials, Int. J. Plasticity 24 (2008) 55–73.
[16] N. Weck, Maxwell's boundary value problems on Riemannian manifolds with nonsmooth boundaries, J. Math. Anal. Appl. 46 (1974) 410–437.