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Numerical Analysis

Spectral stability of finite volume schemes for linear hyperbolic systems

Stabilité spectrale des schémas volumes finis pour les systèmes hyperboliques linéaires

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A B S T R A C T

In this Note we prove the spectral stability of a large class of finite volume schemes applied to hyperbolic systems of linear partial differential equations on multidimensional unstructured meshes. This class requires that the upwinding matrix has positive eigenvalues and is codiagonalisable with the system matrices. That includes among others the upwind and centred implicit schemes, and the upwind explicit scheme under a CFL condition.

R É S U M É

Dans cette Note nous démontrons la stabilité spectrale d'une grande classe de schémas volumes finis pour la résolution des systèmes hyperboliques d'équations aux dérivées partielles linéaires sur maillages non structurés. Cette classe requiert que la matrice de décentrement ait des valeurs propres positives et soit codiagonalisable avec les matrices du système. Elle inclut notament les schémas centré et décentré amont implicites, et le schéma décentré amont explicite sous une condition CFL.

Version française abrégée

Nous nous intéressons dans cette Note à la stabilité spectrale de la résolution numérique des systèmes hyperboliques linéaires d'équations aux dérivées partielles par des schémas volumes finis implicites et explicites sur des maillages non structurés. L'analyse des problèmes linéaires est à la base de l'étude de systèmes plus complexes tels les modèles de dynamique des fluides compressibles (voir par exemple [6]). Ceux ci sont dotés d'une entropie mathématique qui permet de se ramener à un système symétrique, dit de Friedrichs.

Les schémas implicites, permettent en pratique l'utilisation d'un pas de temps non contraint par la condition CFL, ainsi que l'utilisation de flux non décentrés, qui peuvent se montrer très utiles pour la simulation des écoulements à faibre nombre de Mach (voir [2]). Bien qu'utilisés de longue date avec succès dans de nombreux domaines d'application (voir par exemple [1]), peu de résultats existent concernant la convergence des solutions capturées par ces schémas volumes finis pour des systèmes d'équations multidimensionels. En particulier le rôle joué par le décentrement dans le cadre implicite est peu abordé à notre connaissance. [9] démontre la convergence pour la norme L^2 des schémas décentrés amont explicite et implicite dans un domaine infini. Récemment dans [4] les auteurs montrent la stabilité spectrale du schéma amont pour le cas de l'équation d'advection dans un domaine borné. L'utilisation du théorème de Gershgorin pour localiser les valeurs propres dans le demi plan complexe $\mathrm{Re}(z) \leqslant 0$ n'est cependant plus possible dans le cas des systèmes d'équations, et même

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dans le cas scalaire pour un décentrement autre que amont. Dans cette Note nous nous plaçons dans un domaine borné ce qui nous permet d'employer des outils d'algèbre linéaire en dimension finie, pour étudier le spectre de l'opérateur discret pour un système hyperbolique symétrisable (1) et un schéma volumes finis implicite à décentrement quelconque (2).

On étudie donc le spectre de l'opérateur $\mathcal{M}$ du schéma implicite $\mathcal{M}\mathcal{U}^{n+1}=\mathcal{U}^n$ afin d'étudier son inversibilité et le caractère borné de ses puissances inverses. On en déduit une caractérisation des décentrement pour lesquels $\{\mathcal{M}^n, n\in\mathbb{N}\}$ est borné, ce qui implique le caractère borné des solutions discrètes à maillage et pas de temps fixés. Notre résultat principal (Théorème 2.3) énonce que le schéma implicite (2) sur un maillage non structuré quelconque est spectralement stable quelque soit le pas de temps sous réserve que sur chaque interface $f_{\alpha\beta}$ la matrice de décentrement $D(\boldsymbol{\omega}_{\alpha\beta})$ ait des valeurs propres positives et soit codiagonalisable avec $A(\boldsymbol{\omega}_{\alpha\beta})$.

A cette fin on suppose les matrices A_k cosymétrisables, on définit la matrice diagonale $\mathcal{V}$ qui est liée au caractère non structuré du maillage, et les matrices $\mathcal{A}$ et $\mathcal{D}$ (Eq. (3)) qui sont les parties symétrique et antisymétrique de l'opérateur divergence discrète $\mathcal{V}^{-1}\mathcal{M}'$ dans la base associée à la matrice de passage $\mathcal{V}^{\frac{1}{2}}$. Les formes quadratiques associées à $\mathcal{A}$ et $\mathcal{D}$ s'expriment en fonction des formes quadratiques associées aux matrices locales $A(\boldsymbol{\omega}_{\alpha\beta})$ et $D(\boldsymbol{\omega}_{\alpha\beta})$ à chaque interface $f_{\alpha\beta}$ (Lemme 2.1). En supposant que les matrices $A(\boldsymbol{\omega}_{\alpha\beta})$ et $D(\boldsymbol{\omega}_{\alpha\beta})$ commutent à chaque interface, on peut alors borner le spectre de l'opérateur $\mathcal{V}^{-1}\mathcal{M}'$ et montrer que toute valeur propre imaginaire pure est défective (Lemme 2.2). Enfin on montre sous réserve que $D(\boldsymbol{\omega}_{\alpha\beta})$ ait toutes ses valeurs propres positives, que la matrice $\mathcal{M}$ du schéma a des valeurs propres soit égales à 1, soit de module strictement plus grands que 1 (Théorème 2.3). On en conclut alors que les schémas implicite et de Crank–Nicolson sont spectralement stables pour tout pas de temps. Le schéma explicite requiert un décentrement non nul $D\geqslant r|A|$ et un pas de temps soumis à une condition CFL. Les hypothèses du Théorème 2.3 permettent de déduire par exemple la stabilité spectrale des schémas implicites et de Crank–Nicolson classiques qu'ils soient centrés ($D=0$), décentrés amont ($D=|A|$), ou de Rusanov ($D=\rho_A\mathbb{I}_{mN}$), et des schémas explicites avec décentrement non nul sous condition CFL.

1. Introduction

Hyperbolic systems of partial differential equations arise from the modelling of various physical phenomena such as fluid motion, hyperelasticity, traffic or wave modelling. Various finite volume schemes are used for their numerical simulation on complex multidimensional geometries with an unstructured mesh (see for example [8]). In that context, implicit schemes have been used for a long time to solve large multidimensional problems (see the review article [1]) whether with some flux upwinding or not [2,1]. However, little theoretical results exist regarding their convergence when the scheme is not upwind (see [3,9,4]). In this paper we address the question of their spectral stability in the linear symmetrisable case for an abstract upwinding matrix.

We consider a closed d-dimensional manifold Ω, and seek for a vector field $U(\boldsymbol{x},t)\in\mathbb{R}^m$ with $\boldsymbol{x}\in\Omega$, $t\in\mathbb{R}^+$, satisfying the following linear system of conservation laws

$$\frac{\partial U}{\partial t}(\boldsymbol{x},t)+\nabla.F(U)(\boldsymbol{x},t)=0, \tag{1}$$

where A_k are $m\times m$ real matrices and $F(U)=(A_1U,\dots,A_dU)$ is a linear flux function.

If there exists a symmetric positive definite matrix E such that EA_k is symmetric for all k, then the Cauchy problem for system (1) is well posed in $L^2(\Omega)^m$ (see [7]). In Lemma 2.2 and Theorem 2.3 we will make the assumption that the matrices A_k are cosymmetrisable and thus for any vector $\boldsymbol{\omega}=(\omega_1,\dots,\omega_d)\in\mathbb{R}^d$ the matrix $A(\boldsymbol{\omega})=\sum_{k=1}^d\omega_kA_k$ will be diagonalisable with real eigenvalues.

In order to approximate numerically the solutions of system (1), Ω is partitioned or approximated by N polyhedral cells C_α with measure $v_\alpha>0$. Two neighbouring cells C_α and C_β are separated by an interface $f_{\alpha\beta}$ with an associated unit normal vector $\boldsymbol{\omega}_{\alpha\beta}\in\mathbb{R}^d$ oriented from C_α toward C_β ($\boldsymbol{\omega}_{\alpha\beta}=-\boldsymbol{\omega}_{\alpha\beta}$), and a measure $s_{\alpha\beta}>0$ ($s_{\alpha\beta}=s_{\beta\alpha}$). The set of neighbours of a cell C_α is denoted $\nu(\alpha)$ and the mesh is assumed to have no boundary i.e. $\forall\alpha$, $\sum_{\beta\in\nu(\alpha)}s_{\alpha\beta}\boldsymbol{\omega}_{\alpha\beta}=0$. Using the finite volume framework [5], the discrete unknown U_α^n approximates the average value of the unknown U in the cell C_α at time $n\Delta t$ where $\Delta t>0$ is the time step. Assuming the initial data $U_\alpha^0, \alpha=1,\dots,N$ are known, an implicit scheme for the resolution of (1) is given by the following induction equations

$$\frac{U_\alpha^{n+1}-U_\alpha^n}{\Delta t}-\sum_{\beta\in\nu(\alpha)}\frac{s_{\alpha\beta}}{v_\alpha}A^-(\boldsymbol{\omega}_{\alpha\beta})\big(U_\alpha^{n+1}-U_\beta^{n+1}\big)=0, \tag{2}$$

where $A^-=\frac{A-D}{2}$, D is the upwinding matrix of the scheme, requested to satisfy $D(\boldsymbol{\omega}_{\alpha\beta})=D(\boldsymbol{\omega}_{\beta\alpha})$. Typical examples are $D=|A|$ the matrix absolute value for the upwind scheme, and $D=0$ for the centred scheme, or $D=\rho_A\mathbb{I}_m$ for the Rusanov scheme.

Defining $\mathcal{U}^n={}^t(U_1^n,\dots,U_N^n)$, scheme (2) takes the matrix form $\mathcal{M}\mathcal{U}^{n+1}=\mathcal{U}^n$ and is said to be spectrally stable if $\mathcal{M}$ is invertible and its powers $\mathcal{M}^{-n}, n\in\mathbb{N}$ form a bounded set. In order to deduce spectral stability, it is sufficient to prove that any eigenvalue λ of $\mathcal{M}^{-1}$ satisfies $|\lambda|<1$ or is defective (see [4]).

2. Spectral stability theorems

In order to study the spectral stability of scheme (2), we introduce $\mathbb{I}_k$ the size k identity matrix, the maximal spectral radii $\rho_A = \max_{f_{\alpha\beta}}\{\rho(A(\boldsymbol{\omega}_{\alpha\beta}))\}$ and $\rho_D = \max_{f_{\alpha\beta}}\{\rho(D(\boldsymbol{\omega}_{\alpha\beta}))\}$, the characteristic length

$$\Delta x = \min_{\alpha}\left\{\frac{v_\alpha}{\sum_{\beta\in\nu(\alpha)} s_{\alpha\beta}}\right\},$$

$\mathcal{V}$ the diagonal matrix made of the N diagonal blocks $\frac{v_\alpha}{\Delta x}\mathbb{I}_m, \alpha = 1,\dots,N$, $\mathcal{A}$ and $\mathcal{D}$ the block matrices made of the N^2 blocks $\mathcal{A}_{\alpha\beta}$ and $\mathcal{D}_{\alpha\beta}$,

$$\mathcal{A}_{\alpha\beta} = \begin{cases} 0 & \text{if } \beta \notin \nu(\alpha)\cup\{\alpha\}, \\ s_{\alpha\beta}A(\boldsymbol{\omega}_{\alpha\beta}) & \text{if } \beta\in\nu(\alpha), \\ 0 & \text{if } \alpha=\beta, \end{cases} \qquad \mathcal{D}_{\alpha\beta} = \begin{cases} 0 & \text{if } \beta \notin \nu(\alpha)\cup\{\alpha\}, \\ -s_{\alpha\beta}D(\boldsymbol{\omega}_{\alpha\beta}) & \text{if } \beta\in\nu(\alpha), \\ \sum_{\gamma\in\nu(\alpha)} s_{\alpha\gamma}D(\boldsymbol{\omega}_{\alpha\gamma}) & \text{if } \alpha=\beta. \end{cases} \tag{3}$$

Defining $\mathcal{M}' = \frac{1}{2}(\mathcal{A}+\mathcal{D})$, scheme (2) can be rewritten $(\mathbb{I}_{mN} + \frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}')\mathcal{U}^{n+1} = \mathcal{U}^n$.

Lemma 2.1 *(Characterisation of $\mathcal{A}$ and $\mathcal{D}$). For any vector $\mathcal{U} = (U_1,\dots,U_N)\in\mathbb{C}^{mN}$*

$$ {}^t\overline{\mathcal{U}}\mathcal{D}\mathcal{U} = \sum_{f_{\alpha\beta}} s_{\alpha\beta}{}^t(\overline{U_\alpha - U_\beta})D(\boldsymbol{\omega}_{\alpha\beta})(U_\alpha - U_\beta), \tag{4}$$

$$ {}^t\overline{\mathcal{U}}\mathcal{A}\mathcal{U} = \sum_{f_{\alpha\beta}} s_{\alpha\beta}{}^t\overline{U}_\alpha A(\boldsymbol{\omega}_{\alpha\beta})U_\beta + s_{\beta\alpha}{}^t\overline{U}_\beta A(\boldsymbol{\omega}_{\beta\alpha})U_\alpha = \sum_{f_{\alpha\beta}} s_{\alpha\beta}{}^t(\overline{U_\alpha + U_\beta})A(\boldsymbol{\omega}_{\alpha\beta})(U_\beta - U_\alpha). \tag{5}$$

Proof. Eq. (4) is a consequence of the following calculations

$$\begin{aligned} {}^t\overline{\mathcal{U}}\mathcal{D}\mathcal{U} &= \sum_{\alpha=1}^N {}^t\overline{U}_\alpha \sum_{\beta\in\nu(\alpha)} s_{\alpha\beta}D(\boldsymbol{\omega}_{\alpha\beta})(U_\alpha - U_\beta) \\ &= \sum_{f_{\alpha\beta}} s_{\alpha\beta}{}^t\overline{U}_\alpha D(\boldsymbol{\omega}_{\alpha\beta})(U_\alpha - U_\beta) + s_{\beta\alpha}{}^t\overline{U}_\beta D(\boldsymbol{\omega}_{\beta\alpha})(U_\beta - U_\alpha). \end{aligned}$$

We obtain the first equality in Eq. (5) from

$$\begin{aligned} {}^t\overline{\mathcal{U}}\mathcal{A}\mathcal{U} &= \sum_{\alpha=1}^N {}^t\overline{U}_\alpha \sum_{\beta\in\nu(\alpha)} s_{\alpha\beta}A(\boldsymbol{\omega}_{\alpha\beta})U_\beta \\ &= \sum_{f_{\alpha\beta}} s_{\alpha\beta}{}^t\overline{U}_\alpha A(\boldsymbol{\omega}_{\alpha\beta})U_\beta + s_{\beta\alpha}{}^t\overline{U}_\beta A(\boldsymbol{\omega}_{\beta\alpha})U_\alpha. \end{aligned}$$

The second equality in (5) is a consequence of $\sum_{\beta\in\nu(\alpha)} {}^t\overline{U}_\alpha A(\boldsymbol{\omega}_{\alpha\beta})U_\alpha = \sum_{\alpha\in\nu(\beta)} {}^t\overline{U}_\beta A(\boldsymbol{\omega}_{\alpha\beta})U_\beta = 0$. □

Lemma 2.2 *(Spectral properties of $\mathcal{V}^{-1}\mathcal{M}'$). Assume that the matrices A_k are cosymmetrisable and $\forall f_{\alpha\beta}$, $D(\boldsymbol{\omega}_{\alpha\beta})$ is codiagonalisable with $A(\boldsymbol{\omega}_{\alpha\beta})$. Then any eigenvalue λ' of $\mathcal{V}^{-1}\mathcal{M}'$ satisfies $|\mathrm{Re}(\lambda')| \leqslant \rho_D$ and $|\mathrm{Im}(\lambda')| \leqslant \frac{1}{2}\rho_A$. Moreover if $\forall f_{\alpha\beta}$, $D(\boldsymbol{\omega}_{\alpha\beta})$ has positive eigenvalues then $\mathrm{Re}(\lambda') \geqslant 0$ and*

- *$\mathrm{Re}(\lambda') = 0$ implies that λ' is a non-defective eigenvalue.*
- *if $\exists r > 0$ s.t. $\forall f_{\alpha\beta}$, $|A(\boldsymbol{\omega}_{\alpha\beta})| \leqslant rD(\boldsymbol{\omega}_{\alpha\beta})$ then $|\mathrm{Im}(\lambda')|^2 \leqslant r^2\rho_D|\mathrm{Re}(\lambda')|$.*

Proof. The fact that $A(\boldsymbol{\omega}_{\alpha\beta})$ and $D(\boldsymbol{\omega}_{\alpha\beta})$ are codiagonalisable implies that any change of base that symmetrises $A(\boldsymbol{\omega}_{\alpha\beta})$ symmetrises $D(\boldsymbol{\omega}_{\alpha\beta})$. Hence as all the matrices A_k are assumed cosymmetrisable, up to a change of base all the matrices $A(\boldsymbol{\omega}_{\alpha\beta})$ and $D(\boldsymbol{\omega}_{\alpha\beta})$, can be considered symmetric. $\mathcal{A}$ is then antisymmetric and $\mathcal{D}$ is symmetric.

As $\mathcal{V}^{-1}\mathcal{M}' = \mathcal{V}^{-\frac{1}{2}}(\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}})\mathcal{V}^{\frac{1}{2}}$, $\mathcal{V}^{-1}\mathcal{M}'$ is similar to $\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}}$. Let $\mathcal{U}$ be a unitary eigenvector of $\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}}$ associated to the eigenvalue λ'. As $\mathcal{M}' = \frac{1}{2}(\mathcal{A}+\mathcal{D})$, we have $\mathrm{Re}(\lambda') = \frac{1}{2}{}^t\overline{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}\mathcal{U}$ and $\mathrm{Im}(\lambda') = \frac{-i}{2}{}^t\overline{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{A}\mathcal{V}^{-\frac{1}{2}}\mathcal{U}$. From (4) we deduce the following inequalities:

$$\left|{}^t\bar{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}\mathcal{U}\right| \leqslant \rho_D \Delta x \sum_{f_{\alpha\beta}} s_{\alpha\beta} \left\| \frac{1}{\sqrt{v_\alpha}} U_\alpha - \frac{1}{\sqrt{v_\beta}} U_\beta \right\|_2^2 \tag{6}$$

$$\leqslant \rho_D \Delta x \sum_{f_{\alpha\beta}} 2 s_{\alpha\beta} \left(\frac{1}{v_\alpha} \|U_\alpha\|_2^2 + \frac{1}{v_\beta} \|U_\beta\|_2^2 \right)$$

$$= 2\rho_D \sum_{\alpha=1}^{N} \Delta x \frac{\sum_{\beta \in \nu(\alpha)} s_{\alpha\beta}}{v_\alpha} \|U_\alpha\|_2^2$$

$$\leqslant 2\rho_D. \tag{7}$$

The first equality in (5) gives

$$\left|{}^t\bar{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{A}\mathcal{V}^{-\frac{1}{2}}\mathcal{U}\right| \leqslant \Delta x \sum_{f_{\alpha\beta}} 2 s_{\alpha\beta} \rho_A \left\| \frac{1}{\sqrt{v_\alpha}} U_\alpha \right\|_2 \left\| \frac{1}{\sqrt{v_\beta}} U_\beta \right\|_2$$

$$\leqslant \rho_A \Delta x \sum_{f_{\alpha\beta}} s_{\alpha\beta} \left(\frac{1}{v_\alpha} \|U_\alpha\|_2^2 + \frac{1}{v_\beta} \|U_\beta\|_2^2 \right)$$

$$\leqslant \rho_A,$$

hence $|\mathrm{Re}(\lambda')| \leqslant \rho_D$ and $|\mathrm{Im}(\lambda')| \leqslant \frac{1}{2}\rho_A$. Now if $D(\boldsymbol{\omega}_{\alpha\beta})$ is assumed positive we obtain

$$\mathrm{Re}(\lambda') = \frac{1}{2} {}^t\bar{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}\mathcal{U} = \frac{\Delta x}{2} \sum_{f_{\alpha\beta}} s_{\alpha\beta} {}^t\left(\frac{1}{v_\alpha} \bar{U}_\alpha - \frac{1}{v_\beta} \bar{U}_\beta \right) D(\boldsymbol{\omega}_{\alpha\beta}) \left(\frac{1}{v_\alpha} U_\alpha - \frac{1}{v_\beta} U_\beta \right) \geqslant 0.$$

If we furthermore assume that $\mathrm{Re}(\lambda') = 0$, then as $\mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}$ is symmetric positive, ${}^t\bar{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}\mathcal{U} = 0 \Leftrightarrow \mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}\mathcal{U} = 0$. Consequently $\mathcal{U}$ is an eigenvector of both $\mathcal{V}^{-\frac{1}{2}}\mathcal{A}\mathcal{V}^{-\frac{1}{2}}$ (eigenvalue λ') and $\mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}$ (eigenvalue 0), and therefore of both $\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}}$ and its transposed. Assume that the Jordan block associated to λ' is non-trivial. Then there exist $\mathcal{W} \in \mathbb{C}^{mN}$ such that $\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}}\mathcal{W} = \lambda'\mathcal{W} + \mathcal{U}$ and then

$$\begin{aligned}
&{}^t\bar{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}}\mathcal{W} = \lambda'{}^t\bar{\mathcal{U}}\mathcal{W} + {}^t\bar{\mathcal{U}}\mathcal{U} \quad \Rightarrow \quad -2\lambda'{}^t\bar{\mathcal{U}}\mathcal{W} = {}^t\bar{\mathcal{U}}\mathcal{U},\\
&{}^t\overline{\mathcal{W}}\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}}\mathcal{W} = \lambda'{}^t\overline{\mathcal{W}}\mathcal{W} + {}^t\overline{\mathcal{W}}\mathcal{U} \quad \Rightarrow \quad 2\lambda'{}^t\overline{\mathcal{W}}\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}}\mathcal{W} = 2\lambda'^{2t}\overline{\mathcal{W}}\mathcal{W} + {}^t\bar{\mathcal{U}}\mathcal{U}.
\end{aligned}$$

We deduce that $\mathrm{Re}({}^t\overline{\mathcal{W}}\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}}\mathcal{W}) = {}^t\overline{\mathcal{W}}\mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}\mathcal{W} = 0$, and consequently $\mathcal{V}^{-\frac{1}{2}}\mathcal{A}\mathcal{V}^{-\frac{1}{2}}\mathcal{W} = \lambda'\mathcal{W} + \mathcal{U}$, which would imply that $\mathcal{A}$ has a non-trivial Jordan block. That is impossible since $\mathcal{A}$ is antisymmetric hence a diagonalisable matrix. Hence λ' cannot be a defective eigenvalue.

Using Cauchy–Schwarz inequality, the second equality in (5) yields

$$\left|{}^t\bar{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{A}\mathcal{V}^{-\frac{1}{2}}\mathcal{U}\right| \leqslant \Delta x \left(\sum_{f_{\alpha\beta}} s_{\alpha\beta} \left\| \frac{1}{\sqrt{v_\alpha}} U_\alpha + \frac{1}{\sqrt{v_\beta}} U_\beta \right\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{f_{\alpha\beta}} s_{\alpha\beta} \left\| A(\boldsymbol{\omega}_{\alpha\beta}) \left(\frac{1}{\sqrt{v_\beta}} U_\beta - \frac{1}{\sqrt{v_\alpha}} U_\alpha \right) \right\|_2^2 \right)^{\frac{1}{2}}. \tag{8}$$

Now as can be seen from Eqs. (6)–(7), the second term of the product can be bounded by $\sqrt{\frac{2}{\Delta x}}$. Assuming $\forall f_{\alpha\beta},\ |A(\boldsymbol{\omega}_{\alpha\beta})| \leqslant r D(\boldsymbol{\omega}_{\alpha\beta})$ we deduce $\forall U \in \mathbb{C}^m,\ ||A(\boldsymbol{\omega}_{\alpha\beta})U||_2^2 \leqslant r^2 \rho_D {}^t U D U$ and the last term in (8) can be bounded by $\sqrt{\frac{r^2}{\Delta x}\rho_D {}^t\bar{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}\mathcal{U}}$. We eventually obtain

$$\left|{}^t\bar{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{A}\mathcal{V}^{-\frac{1}{2}}\mathcal{U}\right| \leqslant \sqrt{2 r^2 \rho_D {}^t\bar{\mathcal{U}}\mathcal{V}^{-\frac{1}{2}}\mathcal{D}\mathcal{V}^{-\frac{1}{2}}\mathcal{U}},$$

hence $|\mathrm{Im}(\lambda')|^2 \leqslant r^2 \rho_D |\mathrm{Re}(\lambda')|$. □

Theorem 2.3 *(Spectral stability of implicit and explicit schemes). Assume that the matrices A_k are cosymmetrisable and $\forall f_{\alpha\beta},\ D(\boldsymbol{\omega}_{\alpha\beta})$ is codiagonalisable with $A(\boldsymbol{\omega}_{\alpha\beta})$ and has positive eigenvalues. Then*

- *the implicit scheme* (2) *is spectrally stable for any time step Δt.*
- *the corresponding explicit scheme $\mathcal{U}^{n+1} = (\mathbb{I}_{mN} - \frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}')\mathcal{U}^n$ is spectrally stable provided $\frac{\Delta t}{\Delta x} < \frac{2}{(1+r^2)\rho_D}$ with $r > 0$ such that $\forall f_{\alpha\beta},\ |A(\boldsymbol{\omega}_{\alpha\beta})| \leqslant r D(\boldsymbol{\omega}_{\alpha\beta})$.*

– *the corresponding Crank–Nicolson scheme*

$$\left(\mathbb{I}_{mN}+\frac{1}{2}\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}'\right)\mathcal{U}^{n+1}=\left(\mathbb{I}_{mN}-\frac{1}{2}\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}'\right)\mathcal{U}^{n}$$

is spectrally stable for any time step Δt.

Proof. As $\mathcal{V}^{-1}\mathcal{M}'$ is similar to $\mathcal{V}^{-\frac{1}{2}}\mathcal{M}'\mathcal{V}^{-\frac{1}{2}}$ with eigenvalues λ' satisfying $\mathrm{Re}(\lambda')\geqslant 0$ (Lemma 2.2), we deduce that $\mathcal{M}=\mathbb{I}_{mN}+\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}'$ has eigenvalues $\lambda=1+\frac{\Delta t}{\Delta x}\lambda'$ with either $\lambda=1$ or $|\lambda|>1$. $\mathcal{M}$ is therefore invertible, and the eigenvalue 1 which corresponds to the case $\mathrm{Re}(\lambda')=0$ is not defective according to Lemma 2.2. We conclude that the powers of $\mathcal{M}^{-1}$ are bounded and the implicit scheme is spectrally stable.

Similarly, as 1 is a non-defective eigenvalue of $(\mathbb{I}_{mN}-\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}')$, in order to prove the spectral stability of the explicit scheme, it is sufficient to prove that any eigenvalue $\lambda=1-\frac{\Delta t}{\Delta x}\lambda'$ associated to $\lambda'\neq 0$ has modulus strictly smaller than 1. However,

$$\left|1-\frac{\Delta t}{\Delta x}\lambda'\right|<1\quad\Leftrightarrow\quad 0<\frac{\Delta t}{\Delta x}<\frac{2\mathrm{Re}(\lambda')}{|\lambda'|^2},$$

and as Lemma 2.2 implies that $\frac{\mathrm{Re}(\lambda')}{|\lambda'|^2}\geqslant\frac{1}{(1+r^2)\rho_D}$, we conclude that the explicit scheme is spectrally stable whenever $0\leqslant\frac{\Delta t}{\Delta x}<\frac{2}{(1+r^2)\rho_D}$.

Similarly to the implicit scheme matrix $\mathcal{M}$, the matrix $(\mathbb{I}_{mN}+\frac{\Delta t}{2}\mathcal{V}^{-1}\mathcal{M}')$ is invertible and the Crank–Nicolson scheme can be rewritten

$$\mathcal{U}^{n+1}=\left(\mathbb{I}_{mN}+\frac{1}{2}\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}'\right)^{-1}\left(\mathbb{I}_{mN}-\frac{1}{2}\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}'\right)\mathcal{U}^{n}.$$

The matrices $(\mathbb{I}_{mN}+\frac{1}{2}\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}')^{-1}$ and $\mathbb{I}_{mN}-\frac{1}{2}\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}'$ share the same eigenvectors, hence the eigenvalues of their product have the form $\frac{1-\frac{1}{2}\frac{\Delta t}{\Delta x}\lambda'}{1+\frac{1}{2}\frac{\Delta t}{\Delta x}\lambda'}$ where λ' is an eigenvalue of $\mathcal{V}^{-1}\mathcal{M}'$. If $\mathrm{Re}(\lambda')>0$ then $|\frac{1-\frac{1}{2}\frac{\Delta t}{\Delta x}\lambda'}{1+\frac{1}{2}\frac{\Delta t}{\Delta x}\lambda'}|<1$. If $\mathrm{Re}(\lambda')=0$ then $|\frac{1-\frac{1}{2}\frac{\Delta t}{\Delta x}\lambda'}{1+\frac{1}{2}\frac{\Delta t}{\Delta x}\lambda'}|=1$, but again as purely imaginary eigenvalues of $\mathcal{V}^{-1}\mathcal{M}'$ are non-defective (Lemma 2.2), unitary eigenvalues of $(\mathbb{I}_{mN}+\frac{1}{2}\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}')^{-1}(\mathbb{I}_{mN}-\frac{1}{2}\frac{\Delta t}{\Delta x}\mathcal{V}^{-1}\mathcal{M}')$ are non-defective. The Crank–Nicolson scheme is therefore spectrally stable for any time step Δt. □

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