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Partial Differential Equations

Weak solutions to the incompressible Euler equations with vortex sheet initial data

Solutions faibles des équations d'Euler incompressibles avec nappe de tourbillon comme donnée initiale

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ABSTRACT

We construct infinitely many admissible weak solutions to the incompressible Euler equations with initial data given by the classical vortex sheet. The construction is based on the method introduced recently in De Lellis and Székelyhidi Jr. (2009, 2010) [2,3] using convex integration. In particular, the vorticity is not a bounded measure. Instead, the energy decreases in time due to a linearly expanding turbulent zone around the vortex sheet.

RÉSUMÉ

Nous construisons une infinité de solutions faibles admissibles des équations d'Euler incompressibles avec nappes de tourbillons classiques pour données initiales. La construction repose sur la méthode introduite récemment dans De Lellis et Székelyhidi Jr. (2009, 2010) [2,3] faisant appel à l'intégration convexe. En particulier, la vorticité n'est pas une mesure bornée. Au lieu de cela, l'énergie décroît en temps, à cause d'une zone turbulente, entourant la nappe de tourbillon et augmentant linéairement en temps.

1. Introduction

In [2,3] an approach towards the construction of weak solutions of the incompressible Euler equations was introduced, based on convex integration and Baire category arguments, in order to recover the celebrated non-uniqueness results of Scheffer [6] and Shnirelman [7]. Furthermore, a strategy, based on the notion of subsolution, was introduced, which can be used to construct weak solutions to the initial value problem with several additional properties. In particular this strategy was recently used by E. Wiedemann to give a general global existence (and non-uniqueness) result for weak solutions [9].

On the other hand it is well known that weak solutions of the incompressible Euler equations satisfy the so-called weak-strong uniqueness property. This can be stated as follows. We call a weak solution *admissible* if it satisfies the weak energy inequality

$$\int \left|v(x,t)\right|^2 \mathrm{d}x \leqslant \int \left|v_0(x)\right|^2 \mathrm{d}x \quad \text{for a.e. } t \geqslant 0,$$

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where v_0 is the initial data. If there exists a (strong) solution w to the same initial data v_0 such that $\int_0^T \|\nabla w + \nabla w^t\|_{L^\infty}\,dt < \infty$, then w is unique in the class of weak admissible solutions. See [5,1] for details. The apparent contradiction of this statement with the global non-uniqueness result of Wiedemann is resolved, since those weak solutions are in general not admissible. On the other hand admissibility by itself does not imply uniqueness, as was shown in [3]. In other words there exist initial data v_0, for which there exist infinitely many distinct admissible weak solutions. Such initial data are called *wild initial data* in [3]. Subsequently, in [8] it was proved that the set of wild initial data is dense in solenoidal L^2 vector-fields with respect to the strong L^2 topology.

Obviously, wild initial data have to possess a certain amount of irregularity. This follows from the weak-strong uniqueness and classical local existence results. From the construction of wild initial data in [3,8] it is not clear how bad this irregularity needs to be. In this short note we show that the classical vortex-sheet with a flat interface is a wild initial data. More precisely, we fix the domain to be the n-dimensional torus $\mathbb{T}^n$ with side length 1 and v_0 to be the periodic extension of

$$v_0(x) = \begin{cases} e_1 & \text{if } x_n \in (0, \frac{1}{2}), \\ -e_1 & \text{if } x_n \in (-\frac{1}{2}, 0). \end{cases} \tag{1}$$

In other words let $v_0(x) := s(t)e_1$, where $s(t) = \operatorname{sign}(\sin(2\pi t))$.

Theorem 1.1. *There exist infinitely many admissible weak solutions $v \in L^\infty(\mathbb{T}^n \times (0,\infty))$ with initial data v_0. Infinitely many among these satisfy the energy equality, and infinitely many have strictly decreasing energy in time.*

In two dimensions weak solutions to vortex sheet initial data, i.e. v_0 with

$$\operatorname{curl} v_0 \in \mathcal{M}_+\big(\mathbb{T}^2\big) \cap H^{-1}\big(\mathbb{T}^2\big), \tag{2}$$

where $\mathcal{M}_+$ stands for non-negative Radon measures of finite mass, have been constructed in the celebrated paper [4] by Delort. By the very construction (passing to the weak limit from smooth solutions of Euler), the solutions of Delort are admissible. There is a fundamental difference, however, between the solutions of Theorem 1.1 and the solutions of Delort: even though v_0 in (1) satisfies (2), in general $\operatorname{curl} v$ will not be a bounded measure for any positive time $t > 0$, due to high-frequency oscillations in the velocity v on a set of positive Lebesgue measure.

In order to prove the theorem we recall the following result from [3]. To start with, recall the definition of subsolution. To this end let us fix a non-negative function $\bar{e} \in L^1_{loc}(\mathbb{T}^n \times (0,T))$.

Definition 1.2 *(Subsolutions).* A subsolution to the incompressible Euler equations with respect to the kinetic energy density $\bar{e}$ is a triple $(\bar{v}, \bar{u}, \bar{q}) : \mathbb{T}^n \times (0,T) \to \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R}$ with $\bar{v} \in L^2_{loc}$, $\bar{u} \in L^1_{loc}$, $\bar{q} \in \mathcal{D}'$, such that

$$\begin{cases} \partial_t \bar{v} + \operatorname{div} \bar{u} + \nabla \bar{q} = 0 \\ \operatorname{div} \bar{v} = 0, \end{cases} \quad \text{in the sense of distributions;} \tag{3}$$

and

$$\bar{v} \otimes \bar{v} - \bar{u} \leqslant \frac{2}{n}\bar{e} I \quad \text{a.e.} \tag{4}$$

Here $\mathcal{S}_0^{n\times n}$ denotes the set of symmetric traceless $n \times n$ matrices and I is the identity matrix. Observe that subsolutions automatically satisfy $\frac{1}{2}|\bar{v}|^2 \leqslant \bar{e}$ a.e. If in addition (4) is an equality a.e. then $\bar{v}$ is a weak solution of the Euler equations.

Theorem 1.3. *(See Proposition 2 in [3].) Let $(\bar{v}, \bar{u}, \bar{q})$ be a subsolution with respect to $\bar{e}$ on $\mathbb{T}^n \times (0,T)$. Assume that $\mathcal{U} \subset \mathbb{T}^n \times (0,T)$ is an open subset such that $(\bar{v}, \bar{u}, \bar{q})$ and $\bar{e}$ are continuous on $\mathcal{U}$ and*

$$\bar{v} \otimes \bar{v} - \bar{u} < \frac{2}{n}\bar{e} \quad \text{on } \mathcal{U}. \tag{5}$$

Then there exist infinitely many $(\tilde{v}, \tilde{u}) \in L^\infty_{loc}(\mathbb{T}^n \times \mathbb{R})$ with $\tilde{v} \in C(\mathbb{R}; L^2_{weak}(\mathbb{T}^n))$ such that $(\tilde{v}, \tilde{u}, 0)$ satisfies (3), $(\tilde{v}, \tilde{u}) = 0$ a.e. in $\mathcal{U}^c$, and

$$(\bar{v} + \tilde{v}) \otimes (\bar{v} + \tilde{v}) - (\bar{u} + \tilde{u}) = \frac{2}{n}\bar{e} I \quad \text{a.e. in } \mathcal{U}. \tag{6}$$

In particular, if $v := \bar{v} + \tilde{v}$ in the above theorem, then v satisfies the incompressible Euler equations in the space–time region $\mathcal{U}$ with pressure p, where

$$\frac{1}{2}|v|^2 = \bar{e}, \qquad p = \bar{q} - \frac{2}{n}\bar{e}. \tag{7}$$

In other words, if the subsolution is a continuous, strict subsolution in some subregion, then it is possible (in a highly non-unique way) to add a perturbation so that the sum is a solution of the Euler equations with prescribed kinetic energy density. This theorem is essentially the content of Proposition 2 in [3] with an almost identical proof, except that in the proof one needs to perform the covering inside the region $\mathcal{U}$ rather than in all of space–time. Theorem 1.3 leads to the following criterion for wild initial data.

Theorem 1.4. *Let $v_0 \in L^2(\mathbb{T}^n)$ with $\operatorname{div} v_0 = 0$. Assume that there exists $T > 0$ and $\bar{e} \in C([0,T]; L^1(\mathbb{T}^n))$ with*

$$\int \bar{e}(x,t)\,\mathrm{d}x \leqslant \int \frac{1}{2}\left|v_0(x)\right|^2 \mathrm{d}x \quad \text{for all } t \geqslant 0$$

and there exists a subsolution $(\bar{v}, \bar{u}, \bar{q})$ with respect to $\bar{e}$ on $\mathbb{T}^n \times [0,T]$ with

$$\bar{v} \in C\big([0,T]; L^2_{weak}\big(\mathbb{T}^n\big)\big), \quad \bar{v}(0) = v_0.$$

Furthermore, assume that there exists an open set $\mathcal{U} \subset \mathbb{T}^n \times (0,T)$ such that $(\bar{v}, \bar{u}, \bar{q})$ and $\bar{e}$ are continuous on $\mathcal{U}$ and

$$\begin{aligned} \bar{v} \otimes \bar{v} - \bar{u} &< \frac{2}{n}\bar{e}I \quad \text{in } \mathcal{U}, \\ \bar{v} \otimes \bar{v} - \bar{u} &= \frac{2}{n}\bar{e}I \quad \text{a.e. } \mathcal{U}^c. \end{aligned} \tag{8}$$

Then there exist infinitely many admissible weak solutions to the incompressible Euler equations on $\mathbb{T}^n \times (0,T)$ with initial data v_0.

2. Proof of Theorem 1.1

Let us consider for simplicity the case $n = 2$, the higher-dimensional case can be dealt with analogously. We wish to apply Theorem 1.4 to v_0 given in (1). To this end we set

$$\bar{v} := (\alpha, 0), \qquad \bar{u} := \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}, \qquad \bar{q} := \beta$$

where $\alpha = \alpha(x_2,t)$, $\beta = \beta(x_2,t)$ and $\gamma = \gamma(x_2,t)$ are functions still to be fixed. With these choices the system (3) reduces to

$$\partial_t \alpha + \partial_{x_2}\gamma = 0. \tag{9}$$

Next, for some $\lambda \in (0,1)$ we set $\beta := \frac{1}{2}\alpha^2$ and $\gamma := -\frac{\lambda}{2}(1-\alpha^2)$. With this choice (9) becomes the inviscid Burgers equation $\partial_t\alpha + \frac{\lambda}{2}\partial_{x_2}\alpha^2 = 0$, with (1-periodic) initial data given by $\alpha(x_2,0) = s(x_2)$. Set α to be the (unique) viscosity solution, given by a rarefaction wave with speed λ at $x_2 = 0$ and constant shocks at $x_2 = \pm\frac{1}{2}$ up to time $T = \frac{1}{2\lambda}$, i.e. the 1-periodic extension of $x_2 \mapsto \alpha(x_2,t)$ given by

$$\alpha(x_2,t) := \begin{cases} -1 & -\frac{1}{2} < x_2 < -\lambda t, \\ \frac{x_2}{\lambda t} & -\lambda t < x_2 < \lambda t, \qquad \text{if } t < \frac{1}{2\lambda}, \\ 1 & \lambda t < x_2 < \frac{1}{2}, \end{cases} \tag{10}$$

and $\alpha(x_2,t) := \frac{x_2}{\lambda t}$ for $x_2 \in (-\frac{1}{2}, \frac{1}{2})$ and $t \geqslant \frac{1}{2\lambda}$. In particular, we have $|\alpha| \leqslant 1$ for all (x,t) and $\{|\alpha| < 1\} = \mathcal{U} := \{(x,t)\colon\ |x_2| < \lambda t\}$. Therefore, in order to verify (8) it suffices to set

$$\bar{e} := \frac{1}{2} - \varepsilon\frac{1-\lambda}{2}\big(1-\alpha^2\big) \tag{11}$$

for some $\varepsilon \in [0,1)$. With this choice of $\mathcal{U}$ and $\bar{e}$ the triple $(\bar{v}, \bar{u}, \bar{q})$ satisfies (8) for all time. Therefore, with Theorem 1.4 we conclude the proof.

Remarks. Observe that initially the maximal expansion rate of the *turbulent zone* $\mathcal{U}$ is linear with speed given by $|v_0| = 1$, and is achieved when $\lambda \to 1$. Using (7) we also note that the pressure corresponding to the solutions constructed is – up to a constant – given by $p = \frac{1}{2}(1-\alpha^2)$. In particular the pressure gradient is concentrated in the turbulent zone. The kinetic energy is given by $E(t) = \int \bar{e}\,\mathrm{d}x$. Using (10) and (11) we find the dissipation rate to be

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{2}{3}\varepsilon\lambda(1-\lambda). \tag{12}$$

In particular the dissipation rate is maximized when $\varepsilon \to 1$ and $\lambda = 1/2$.

Finally we note that as $t \to \infty$, the subsolutions constructed in the proof converge to zero. This corresponds to the fully mixed, isotropic state of the fluid.

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