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Global solutions, and their decay properties, of the spherically symmetric $\mathfrak{su}(2)$ − Einstein–Yang–Mills–Higgs equations

Solutions globales et estimations de décroissance pour les équations d'Einstein–Yang–Mills–Higgs en symétrie sphérique

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A R T I C L E   I N F O

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A B S T R A C T

We generalize, to the $\mathfrak{su}(2)$ − Einstein–Yang–Mills–Higgs system, previous results concerning global solutions of the Einstein-scalar field and the Einstein–Maxwell–Higgs equations. The novelty of the present work is at least twofold. For one thing the assumption on the self-interaction potential is improved. For another thing explanation is furnished why the solutions obtained here decay slower than those of self-gravitating massless scalar fields. Actually this latter phenomenon stems from the non-vanishing of the local charge.

R É S U M É

Nous généralisons, aux équations d'Einstein–Yang–Mills–Higgs, des résultats antérieurs concernant les systèmes Einstein-champ scalaire et Einstein–Maxwell–Higgs. L'originalité de notre travail est au moins double. Premièrement l'hypothèse sur le potentiel d'interaction est améliorée. Deuxièmement une explication est donnée du fait que les solutions établies ici décroissent moins vite que celles obtenues dans le cas des équations d'Einstein-champ scalaire. En effet ce dernier phénomène est dû à la non-nullité de la charge locale.

Version française abrégée

Le présent travail a pour objet la résolution globale des équations d'Einstein–Yang–Mills–Higgs (EYMH) en symétrie sphérique et en coordonnées de Bondi (voir (1) pour la forme de la métrique dans ces coordonnées). On suppose que le champ de Yang–Mills est dans la représentation adjointe de $\mathfrak{su}(2)$ tandis que le champ de Higgs est dans la représentation fondamentale de $\mathfrak{su}(2)$ (voir (2)–(3)). Pour atteindre cet objectif, on utilise une adaptation heureuse des outils forgés dans [4,2] pour réduire le problème à celui de la résolution d'un système d'évolution non linéaire de deux équations aux dérivées partielles. Ceci présente des difficultés mathématiques nouvelles par rapport à [4,2] où on avait à traiter une seule équation d'évolution : le nombre de termes à estimer ici est plus élevé et on doit recourir à des arguments mathématiques supplémentaires pour affronter les termes additionnels. Au prix des calculs ardus combinés à de nombreux instruments mathématiques minutieux, on met en oeuvre une méthode de point fixe pour obtenir, sous l'hypothèse plus générale (H) sur le potentiel d'interaction V, un résultat d'existence et d'unicité globale pour le système EYMH (4)–(10). On montre également

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que cette solution, dont l'espace-temps correspondant est complet vers le futur le long des géodésiques temporelles et isotropes, décroît plus lentement que la solution trouvée par Christodoulou [4] pour le système Einstein-champ scalaire. Une auscultation pointilleuse permet de se rendre compte que ce phénomène de ralentissement de la décroissance de la solution est dû à la non-nullité de la charge locale Q via l'estimation (23) du terme $\frac{Q}{r}$. Certaines questions soulevées par Chae [2] trouvent ainsi une réponse. Il serait intéressant, compte tenu de la structure des équations étudiées, de se questionner sur la possibilité (ou l'impossibilité) de l'emploi des techniques de compactification conforme dues à Penrose [7] pour expliquer le taux de décroissance évoqué ci-dessus via un prolongement par continuité à l'infini isotrope conforme. Au total, on obtient une amélioration et une généralisation des résultats de [4,2]. Le Théorème 3.1 du présent travail englobe aussi bien la résolution globale du système EYMH sans potentiel d'interaction (i.e. $V=0$) que celle du système Einstein–Klein–Gordon (en prenant $\psi=0$ ou $\xi=0$, et $V(t)=\frac{t^{p+1}}{p+1}$, $t\geqslant 0$).

1. The spherically symmetric $\mathfrak{su}(2)$ — Einstein–Yang–Mills–Higgs system

We assume that the Yang–Mills field is in the adjoint representation of $\mathfrak{su}(2)$ while the Higgs field is a complex doublet in the fundamental representation of $\mathfrak{su}(2)$. In local coordinates $(x^\alpha)_{\alpha=0,\dots,3}$ on a $4D$ space–time manifold $\mathcal{M}$, and basis $(T_I)_{I=1,2,3}$ of $\mathfrak{su}(2)$, the EYMH equations read (see [3,5,6])

$$R_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}R=T_{\alpha\beta},\qquad g^{\lambda\mu}\widehat{\nabla}_\lambda F^I_{\mu\alpha}=J^I_\alpha,\qquad g^{\lambda\mu}\widehat{\nabla}_\lambda\widehat{\nabla}_\mu\Phi=V'\big(\Phi^\dagger\Phi\big)\Phi.$$

Here $g_{\alpha\beta}$ are the components of the space–time metric. $R_{\alpha\beta}$ and R are, respectively, the Ricci tensor and the scalar curvature of the space–time metric. $F^I_{\alpha\beta}$ are the components of the Yang–Mills strength field F which is related to the Yang–Mills potential A by $F_{\alpha\beta}=\nabla_\alpha A_\beta-\nabla_\beta A_\alpha+[A_\alpha,A_\beta]$, where ∇ denotes the covariant derivative w.r.t. the space–time metric and $[\,,\,]$ denote the Lie brackets of $\mathfrak{su}(2)$. $\widehat{\nabla}_\alpha\Phi$ is the gauge covariant derivative of the complex doublet Higgs field Φ, defined by $\widehat{\nabla}_\alpha\Phi=\nabla_\alpha\Phi-iA^I_\alpha\frac{\sigma_I}{2}\Phi$, where $(\sigma_I)_{I=1,2,3}$ are the Pauli matrices. $\Phi^\dagger$ denotes the hermitian conjugate of Φ. V is a real function defined on $[0,\infty)$, called the self-interaction potential, with derivative V'. $T_{\alpha\beta}$ are the components of the energy–momentum-stress tensor given by

$$T_{\alpha\beta}=g^{\rho\sigma}F_{\alpha\rho}.F_{\beta\sigma}-\frac{1}{4}g_{\alpha\beta}g^{\sigma\lambda}g^{\rho\mu}F_{\sigma\rho}.F_{\lambda\mu}+(\widehat{\nabla}_\alpha\Phi)^\dagger\widehat{\nabla}_\beta\Phi+(\widehat{\nabla}_\beta\Phi)^\dagger\widehat{\nabla}_\alpha\Phi-g_{\alpha\beta}\big(g^{\sigma\rho}(\widehat{\nabla}_\sigma\Phi)^\dagger\widehat{\nabla}_\rho\Phi+V\big(\Phi^\dagger\Phi\big)\big),$$

where the dot denotes the scalar product of $\mathfrak{su}(2)$. J^I_α are the components of the Yang–Mills current given by

$$J^I_\alpha=\Phi^\dagger S^I\widehat{\nabla}_\alpha\Phi-(\widehat{\nabla}_\alpha\Phi)^\dagger S^I\Phi,\quad\text{where } S^I=i\frac{\sigma_I}{2}.$$

We now present the spherically symmetric ansätze and corresponding fields equations. We work in a Bondi coordinates system $(x^\alpha)=(u,r,\theta,\varphi)$ used in [4,2,1]. In this coordinates system the form of the spherically symmetric metric is

$$g_{\alpha\beta}\,dx^\alpha\,dx^\beta=-e^{2\nu}\,du^2-2e^{\nu+\lambda}\,du\,dr+r^2\big(d\theta^2+\sin^2\theta\,d\varphi^2\big),\tag{1}$$

where ν and λ are functions of u and r only. We choose the spherically symmetric Yang–Mills potential of the form

$$A_0=aT_3,\qquad A_1=A_2=A_3=0,\tag{2}$$

where a is a function of u and r only, and $T_3=i\frac{\sigma_3}{2}$. For the spherically symmetric Higgs field we choose the ansatz

$$\Phi=\begin{pmatrix}0\\ \psi+i\xi\end{pmatrix},\tag{3}$$

where ψ and ξ are real functions of u and r only. Writing roughly the EYMH fields equations in the coordinates system (u,r,θ,φ) yields a system of partial differential equations (PDE) which is very difficult to handle. Ahead of overcoming this toughness we introduce, as in [4], a null tetrad $(e_\alpha)_{\alpha=0,\dots,3}$, where $e_0=e^{-\nu}\frac{\partial}{\partial u}-\frac{1}{2}e^{-\lambda}\frac{\partial}{\partial r}$, $e_1=e^{-\lambda}\frac{\partial}{\partial r}$, and (e_2,e_3) is a locally defined orthonormal frame on the unit 2-sphere. After numerous tedious and lengthy calculations we find out from the ansätze (1)–(3) that the Einstein fields equations are equivalent to the following PDE:

$$\begin{aligned}\frac{1}{2}\big(\lambda'+\nu'\big)-2e^{\lambda-\nu}\dot\lambda&=2re^{2(\lambda-\nu)}\bigg[\bigg(\dot\psi-\frac{1}{2}a\xi\bigg)^2+\bigg(\dot\xi+\frac{1}{2}a\psi\bigg)^2\bigg]\\&\quad-2re^{\lambda-\nu}\bigg[\bigg(\dot\psi-\frac{1}{2}a\xi\bigg)\psi'+\bigg(\dot\xi+\frac{1}{2}a\psi\bigg)\xi'\bigg]+\frac{1}{2}r\big[\big(\psi'\big)^2+\big(\xi'\big)^2\big],\end{aligned}\tag{4}$$

$$\nu'-\lambda'+r^{-1}\big(1-e^{2\lambda}\big)+r\bigg(\frac{1}{2}e^{-2\nu}\big(a'\big)^2+e^{2\lambda}V\bigg)=0,\tag{5}$$

$$\lambda'+\nu'=r\big[\big(\psi'\big)^2+\big(\xi'\big)^2\big],\tag{6}$$

$$\nu'' + (\nu' - \lambda')(\nu' + r^{-1}) - e^{\lambda-\nu}\left[(\dot{\lambda})' + (\dot{\nu})' + 2\left(\dot{\psi} - \frac{1}{2}a\xi\right)\psi' + 2\left(\dot{\xi} + \frac{1}{2}a\psi\right)\xi'\right]$$
$$- \frac{1}{2}e^{-2\nu}(a')^2 + \left[(\psi')^2 + (\xi')^2\right] + e^{2\lambda}V = 0, \tag{7}$$

where $\dot{}$ denotes differentiation with respect to u, and $'$ means differentiation with respect to r. The relevant Yang–Mills equations are the following PDE:

$$a'' + (2r^{-1} - \lambda' - \nu')a' - e^{\lambda-\nu}\left((\dot{a})' - \dot{\lambda} - \dot{\nu}\right) = e^{2\lambda}\left[\psi\dot{\xi} - \xi\dot{\psi} + \frac{1}{2}a(\psi^2 + \xi^2)\right], \tag{8}$$

$$a'' + (2r^{-1} - \lambda' - \nu')a' = e^{\lambda+\nu}(\psi\xi' - \xi\psi'). \tag{9}$$

The Higgs equations form the following non-linear evolution system of PDE:

$$2(\dot{\psi})' - a\xi' - \frac{1}{2}a'\xi + 2r^{-1}\left(\dot{\psi} - \frac{1}{2}a\xi\right) - e^{\nu-\lambda}\left[\psi'' + \psi'(2r^{-1} + \nu' - \lambda')\right] = -\psi V' e^{\lambda+\nu},$$
$$2(\dot{\xi})' + a\psi' + \frac{1}{2}a'\psi + 2r^{-1}\left(\dot{\xi} + \frac{1}{2}a\psi\right) - e^{\nu-\lambda}\left[\xi'' + \xi'(2r^{-1} + \nu' - \lambda')\right] = -\xi V' e^{\lambda+\nu}. \tag{10}$$

Remark 1. One can verify by using Bianchi identities that the Einstein equations (5)–(6), together with the YMH equations (8)–(10), are equivalent to the full set of EYMH equations (4)–(10).

2. Reduction of the EYMH equations to a non-linear evolution system

We adapt the tools built up and implemented in [4,2]. Two new functions $h = (r\psi)'$ and $k = (r\xi)'$ are introduced so that

$$\psi = \bar{h} = \frac{1}{r}\int_0^r h(s)\,ds, \qquad \xi = \bar{k} = \frac{1}{r}\int_0^r k(s)\,ds, \qquad \psi' = \frac{h - \bar{h}}{r}, \qquad \xi' = \frac{k - \bar{k}}{r}. \tag{11}$$

The Einstein equation (6) reads

$$\lambda' + \nu' = \frac{1}{r}\left[(h - \bar{h})^2 + (k - \bar{k})^2\right], \tag{12}$$

and the solution of (12) which satisfies the asymptotic condition $\lambda + \nu \to 0$ as $r \to \infty$ is

$$\lambda + \nu = -\int_r^{+\infty} \frac{(h - \bar{h})^2 + (k - \bar{k})^2}{s}\,ds. \tag{12a}$$

Integrating the Yang–Mills equation (9) the solution that exists for all $r \in [0, \infty)$ is given by $a(u, r) = \int_0^r e^{\lambda+\nu}\frac{Q(u,s)}{s^2}\,ds$, where Q is the local charge function defined by $Q(u, r) = \int_0^r s(\bar{h}k - \bar{k}h)(u, s)\,ds$. The Einstein equation (5) then reads

$$(\nu' - \lambda')e^{\nu-\lambda} + r^{-1}e^{\nu-\lambda} = r^{-1}e^{\lambda+\nu} - r\left(\frac{1}{2}e^{-\lambda-\nu}(a')^2 + e^{\lambda+\nu}V\right),$$

which is integrated to give $e^{\nu-\lambda} = \frac{1}{r}\int_0^r e^{\lambda+\nu}[1 - \frac{1}{2}\frac{Q^2}{s^2} - s^2 V]\,ds$. From (10)–(11) we recast the Higgs equations as

$$DW = \frac{1}{2r}(g - \widetilde{g})(W - \overline{W}) - \frac{Q^2 g}{4r^3}(W - \overline{W}) + \frac{Qg}{4r}\sigma_1\overline{W} + \frac{a}{2}i\sigma_2 W - \frac{r}{2}\left[Vg(W - \overline{W}) + V'g\overline{W}\right], \tag{13}$$

where $W = \binom{h}{k}$, $g = e^{\lambda+\nu}$, $\widetilde{g} = e^{\nu-\lambda}$, $D = \frac{\partial}{\partial u} - \frac{\widetilde{g}}{2}\frac{\partial}{\partial r}$.

Remark 2. (i) It is worth noting that the integration of the Einstein equations is achieved under the asymptotic condition $\lambda + \nu \to 0$, $r \to \infty$. The Yang–Mills equation (9) is integrated easily by classical tools (change of unknown function and variation of the constant) to yield the solution $a(u, r)$ that exists for all $r \in [0, +\infty)$.

(ii) For the Yang–Mills field it is easy to see that it vanishes at spatial infinity, i.e. $F(u, r) \to 0$, $r \to +\infty$. Actually, by simple calculation, all the components of the Yang–Mills field F vanish except F_{01} which is given by $F_{01} = -a'T_3$. Using the expression of $a(u, r)$ and estimating the local charge $Q(u, r)$ one easily gains $F_{01}(u, r) \to 0$, $r \to +\infty$.

(iii) It would be of interest to investigate the case of an asymptotic (anti-) de Sitter metric.

3. Existence and uniqueness of global classical solutions

Consider the initial value problem for system (13) with initial datum $W_0(r) = W(0,r)$. Following the works of Chae [2] we define the Banach function spaces $(\mathcal{X}, \|.\|_{\mathcal{X}})$, $(\mathcal{X}_0, \|.\|_{\mathcal{X}_0})$, and $(\mathcal{Y}, \|.\|_{\mathcal{Y}})$ by

$$\mathcal{X} = \{W = W(u,r) \in C^1([0,\infty) \times [0,\infty)) \colon \|W\|_{\mathcal{X}} < \infty\}, \qquad \mathcal{X}_0 = \{v = v(r) \in C^1([0,\infty)) \colon \|v\|_{\mathcal{X}_0} < \infty\},$$
$$\mathcal{Y} = \{W = W(u,r) \in C^1([0,\infty) \times [0,\infty)) \colon W(0,r) = W_0(r),\ \|W\|_{\mathcal{Y}} < \infty\}, \tag{14}$$

where

$$\|W\|_{\mathcal{X}} := \sup_{u,r \geqslant 0} \{(1+r+u)^2 |W(u,r)| + (1+r+u)^3 |W'(u,r)|\},$$
$$\|v\|_{\mathcal{X}_0} := \sup_{r \geqslant 0} \{(1+r)^2 |v(r)| + (1+r)^3 |v'(r)|\}, \qquad \|W\|_{\mathcal{Y}} := \sup_{u,r \geqslant 0} \{(1+r+u)^2 |W(u,r)|\}.$$

As we have mentioned from the outset, the purpose of this paper is to prove the following theorem:

Theorem 3.1. *Assume for the self-interaction potential V that $V \in C^2([0,\infty))$, and there exists a constant $K_0 \geqslant 0$ such that*

$$|V(t)| + t|V'(t)| + t^2|V''(t)| \leqslant K_0 t^{p+1} \quad \forall t \geqslant 0, \quad \text{for some } p \geqslant \frac{3}{2}. \tag{H}$$

Suppose for the initial datum W_0 that

$$W_0 \in C^1([0,\infty)), \quad W_0(r) = O(r^{-2}), \qquad W_0'(r) = O(r^{-3}).$$

Then there exists $d > 0$ such that, for $\|W_0\|_{\mathcal{X}_0} < d$, there exists a unique global classical solution $W \in C^1([0,\infty) \times [0,\infty))$ of (13) satisfying $W(0,r) = W_0(r)$. In addition this solution fulfills the decay estimates

$$|W(u,r)| \leqslant C(1+u+r)^{-2}, \qquad |W'(u,r)| \leqslant C(1+u+r)^{-3}, \tag{15}$$

where $C > 0$ is an affine increasing function of K_0. What's more, the corresponding space–time is time-like and null geodesically complete toward the future.

Proof. We use a contraction method as in [2] whilst improving and correcting at the same time some key estimates therein. We define the mapping $\mathcal{P} : W \mapsto w = \mathcal{P}(W)$, where w is the solution of the first order linear initial value problem

$$Dw = \frac{1}{2r}(g - \tilde{g})(w - \overline{W}) - \frac{Q^2 g}{4r^3}(w - \overline{W}) + \frac{Qg}{4r}\sigma_1 \overline{W} + \frac{a}{2} i\sigma_2 w$$
$$- \frac{r}{2}[Vg(w - \overline{W}) + V'g\overline{W}], \quad w(0,r) = W_0(r). \tag{16}$$

First step: $\mathcal{P}$ is a mapping from a ball of $\mathcal{X}$ into itself. Let $x > 0$ be a real number. B_x denotes the closed ball, in $\mathcal{X}$, of radius x centered at 0. We will prove that x can be chosen small enough such that $\mathcal{P} : B_x \longrightarrow B_x$. The characteristic system of ordinary differential equations associated to the initial value problem (16) is

$$\frac{\mathrm{d}r}{\mathrm{d}u} = -\frac{1}{2}\tilde{g},$$
$$\frac{\mathrm{d}w}{\mathrm{d}u} = \frac{1}{2r}(g - \tilde{g})(w - \overline{W}) - \frac{Q^2 g}{4r^3}(w - \overline{W}) + \frac{Qg}{4r}\sigma_1 \overline{W} + \frac{a}{2} i\sigma_2 w - \frac{r}{2}[Vg(w - \overline{W}) + V'g\overline{W}],$$

with initial data $r(0) = r_0$, $w(0) = W_0$. Suppose $\|W\|_{\mathcal{X}} \leqslant x$. Then after arduous and lengthy calculations, combined with thorough mathematical arguments, we arrive at the following estimate for the solution w of the initial value problem (16):

$$\sup_{r,u \geqslant 0} [(1+u+r)^2 |w(u,r)|] \leqslant \frac{C_5(x^3 + x^5 + x^{2p+1} + x^{2p+3} + \|W_0\|_{\mathcal{X}_0}) \exp(C_4(x^2 + x^4 + x^{2p+2}))}{l^2(x)}, \tag{17}$$

where $l(x) = \exp(-\frac{x^2}{24}) - \frac{x^4}{24} - \frac{K_0 x^{2p+2}}{3}$. Note that the function l has a unique positive root x_0 and $l(x) \in (0,1]$ for all $x \in [0, x_0)$. We also have to estimate $\sup_{r,u \geqslant 0} [(1+u+r)^3 |w'(u,r)|]$. Set $z(u,r) = w'(u,r)$. Differentiation of (16) w.r.t. r gives

$$Dz = N_1 z + B_1 w + B_2 \overline{W} + B_3 \overline{W}', \quad z(0,r_0) = W'(0,r_0), \tag{18}$$

where the matrix functions N_1, B_1, B_2, and B_3 are given by

$$N_1=\left[\frac{\widetilde{g}'}{2}+\frac{g-\widetilde{g}}{2r}-\frac{Q^2g}{4r^3}-\frac{rVg}{2}\right]I_2+\frac{a}{2}i\sigma_2,\qquad B_3=\left[\frac{Q^2g}{4r^3}-\frac{1}{2r}(g-\widetilde{g})+\frac{r(V-V')g}{2}\right]I_2+\frac{Qg}{4r}\sigma_1,$$

$$B_1=\left[\frac{(g-\widetilde{g})'}{2r}-\frac{g-\widetilde{g}}{2r^2}-\frac{QQ'g}{2r^3}+\frac{3Q^2g}{4r^4}-\frac{Q^2g'}{4r^3}-\frac{Vg+r(\Phi^\dagger\Phi)'V'g+rVg'}{2}\right]I_2+\frac{a'}{2}i\sigma_2,$$

$$B_2=\left[\frac{QQ'g}{2r^3}-\frac{3Q^2g}{4r^4}+\frac{Q^2g'}{4r^3}-\frac{(g-\widetilde{g})'}{2r}+\frac{(g-\widetilde{g})}{2r^2}+\frac{Vg+r(\Phi^\dagger\Phi)'V'g+rVg'}{2}\right.$$
$$\left.-\frac{gV'+r[V'g'+(\Phi^\dagger\Phi)'V''g]}{2}\right]I_2+\left(\frac{Q'g+Qg'}{4r}-\frac{Qg}{4r^2}\right)\sigma_1,$$

I_2 being the 2×2 identity matrix. Using similar tools as above we gain the following estimate for the solution z of the initial value problem (18):

$$\sup_{u,r\geqslant 0}\left[(1+r+u)^3\left|z(u,r)\right|\right]$$
$$\leqslant\frac{C_{16}(x^3+x^5+x^{2p+1}+x^{2p+3}+\|W_0\|_{\mathcal{X}_0})(x^2+x^4+x^{2p}+x^{2p+2}+x^{2p+4})\exp(C_{15}(x^2+x^4+x^{2p+2}))}{l^3(x)}.\qquad(19)$$

It follows from (17) and (19) that

$$\|w\|_{\mathcal{X}}$$
$$\leqslant\frac{(x^3+x^5+x^{2p+1}+x^{2p+3}+\|W_0\|_{\mathcal{X}_0})\exp(C_{17}(x^2+x^4+x^{2p+2}))[C_5+C_{16}(x^2+x^4+x^{2p}+x^{2p+2}+x^{2p+4})]}{l^3(x)}.$$

Setting

$$\Omega(x)=\frac{xl^3(x)\exp[-K_7(x^2+x^4+x^{2p+2})]}{C_5+C_{16}(x^2+x^4+x^{2p}+x^{2p+2}+x^{2p+4})}-\left(x^3+x^5+x^{2p+1}+x^{2p+3}\right),$$

one sees that $\Omega(0)=0$, $\Omega'(0)>0$, and $\lim_{x\to+\infty}\Omega(x)=-\infty$. Hence there exists $x_1\in(0,x_0)$ such that the function Ω is strictly monotonically increasing on $[0,x_1]$. We then choose $x\in(0,x_1)$ and $\|W_0\|_{\mathcal{X}_0}<x$ to have $\|w\|_{\mathcal{X}}\leqslant x$.

Second step: the mapping $\mathcal{P}$ contracts in $\mathcal{Y}$. Let w_1 and w_2 be two solutions of (16) with $w_1(0,r)=w_2(0,r)$, $w_j=\mathcal{P}(W_j)$, $W_j\in\mathcal{X}$, $j=1,2$. We use the notations $g_j=g(W_j)$, $Q_j=Q(W_j)$, $V_j=V(W_j)$, $z_j=z(W_j)$, $a_j=a(W_j)$, assume $\max\{\|W_1\|_{\mathcal{X}},\|W_2\|_{\mathcal{X}}\}\leqslant x<x_1$, and set $w_1-w_2=\vartheta$, $W_1-W_2=\Theta$, $\|\Theta\|_{\mathcal{Y}}=y$, $D_1=\frac{\partial}{\partial u}-\frac{\widetilde{g}_1}{2}\frac{\partial}{\partial r}$. Then we have the following non-linear initial value problem with unknown ϑ:

$$D_1\vartheta=N_2\vartheta+B_4\overline{\Theta}+B_5(w_2-\overline{W_2})+B_6,\quad \vartheta(0,r)=0,\qquad(20)$$

where

$$N_2=\left(\frac{g_1-\widetilde{g_1}}{2r}-\frac{Q_1^2g_1}{4r^3}-\frac{rV_1g_1}{2}\right)I_2+\frac{a_1}{2}i\sigma_2,\qquad B_4=\left(\frac{Q_1^2g_1}{4r^3}-\frac{g_1-\widetilde{g_1}}{2r}+\frac{rg_1(V_1-V_1')}{2}\right)I_2+\frac{Q_1g_1}{4r}\sigma_1,$$

$$B_5=\frac{(g_1-\widetilde{g_1})-(g_2-\widetilde{g_2})}{2r}-\frac{(Q_1^2g_1-Q_2^2g_2)}{4r^3}-\frac{r(V_1g_1-V_2g_2)}{2},$$

$$B_6=-\frac{r(V_1'g_1-V_2'g_2)}{2}\overline{W_2}+\frac{Q_1g_1-Q_2g_2}{4r}\sigma_1\overline{W_2}+\frac{\widetilde{g_1}-\widetilde{g_2}}{2}(w_2)'+\frac{a_1-a_2}{2}i\sigma_2w_2.$$

By using repeatedly the mean value theorem and similar calculations as above we gain $\|\vartheta\|_{\mathcal{Y}}\leqslant\Xi(x)y$, where

$$\Xi(x)=\frac{C_{21}(x^2+x^4+x^6+x^{2p}+x^{2p+2}+x^{2p+4})\exp(C_{18}(x^2+x^4+x^{2p+2}))}{l^2(x)}.$$

It is easy to see that the function Ξ is strictly monotonically increasing on $[0,x_1]$ and $\Xi(0)=0$. This shows that there exists $x_2\in(0,x_1]$ such that $\Xi(x)<\frac{1}{2}$ for all $x\in(0,x_2]$. Thus the mapping $W\mapsto\mathcal{P}(W)$ contracts in $\mathcal{Y}$ for $\|W\|_{\mathcal{X}}\leqslant x_2$. This concludes the proof of the global existence and uniqueness of classical solution. The decay properties (15) of the solution are direct consequences of the definition (14) of the Banach spaces $(\mathcal{X},\|\|_{\mathcal{X}})$ and $(\mathcal{Y},\|\|_{\mathcal{Y}})$. Now it is worth noting that, in the course of deriving the estimate (17), one uses among others the estimates

$$\int_0^\infty\frac{|W(u,r)-\overline{W}(u,r)|^2}{r}\,\mathrm{d}r\leqslant\frac{x^2}{24(1+u)^4},\qquad(21)$$

and

$$\left|g(u,r)-\widetilde{g}(u,r)\right| \leqslant \frac{(3+8K_0)(x^2+x^4+x^{2p+2})r^2}{24(1+u)^5(1+u+r)^3}. \tag{22}$$

(12a) and (21)–(22) show that, as $u \to \infty$, the metric given by (1) becomes the Minkowski metric. With this, we are done with the proof of Theorem 3.1. □

Remark 3. (i) Theorem 3.1 was stated and proved, under the more restrictive assumption $p \geqslant 3$, by Chae [2] for the Einstein–Maxwell–Higgs system. Note that the solution obtained here decays slower than that of [4] concerning the spherically symmetric massless Einstein-scalar field system. We found out that this latter fact stems essentially from the estimate

$$\left|\frac{Q}{r}\right| \leqslant \frac{x^2}{2(1+u)^2(1+u+r)}, \tag{23}$$

due to the non-vanishing of the local charge Q. Some questions raised in [2] are thus answered. It would be interesting to find out whether one can use conformal compactification methods of Penrose [7] to explain slow decaying of the solution via extension by continuity to conformal null infinity.

(ii) Theorem 3.1 easily applies so as to encompass global existence and uniqueness of classical spherically symmetric solutions of the EYMH system with vanishing self-interaction potential V and those of the non-linear Einstein–Klein–Gordon system as well. Moreover, in the latter case, it turns out that the solutions possess the same order of decay estimates as those obtained in [4].

(iii) It is worth mentioning that assumption (H) is not fulfilled by the (classical) self-interaction potential $V(t) = t^2$.

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References

[1] D. Chae, Global existence of spherically symmetric solutions to the coupled Einstein and nonlinear Klein–Gordon system, Classical Quantum Gravity 18 (21) (2001) 4589–4605.
[2] D. Chae, Global existence of solutions to the coupled Einstein and Maxwell–Higgs system in the spherical symmetry, Ann. Henri Poincaré 4 (2003) 35–62.
[3] Y. Choquet-Bruhat, Yang–Mills–Higgs fields in three space time dimensions, Mém. Soc. Math. France 46 (1991) 73–97.
[4] D. Christodoulou, The problem of a self-gravitating scalar field, Comm. Math. Phys. 105 (1986) 337–361.
[5] M. Dossa, C. Tadmon, The Goursat problem for the Einstein–Yang–Mills–Higgs system in weighted Sobolev spaces, C. R. Acad. Sci. Paris, Ser. I 348 (2010) 35–39.
[6] M. Dossa, C. Tadmon, The characteristic initial value problem for the Einstein–Yang–Mills–Higgs system in weighted Sobolev spaces, Appl. Math. Res. Express 2010 (2) (2010) 154–231.
[7] R. Penrose, Conformal treatment of infinity, Relativ. Groups Topology (1964) 565–584; republished in Gen. Relativity Gravitation 43 (2011) 901–922.