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Uniform asymptotics for Meixner–Pollaczek polynomials with varying parameters

Analyse asymptotique uniforme des polynômes de Meixner–Pollaczek avec des paramètres variables

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A R T I C L E   I N F O

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A B S T R A C T

In this Note, we study the uniform asymptotics of the Meixner–Pollaczek polynomials $P_n^{(\lambda_n)}(z;\phi)$ with varying parameter $\lambda_n = (n+\frac{1}{2})A$ as $n\to\infty$, where $A>0$ is a constant. Uniform asymptotic expansions in terms of parabolic cylinder functions and elementary functions are obtained for z in two overlapping regions which together cover the whole complex plane.

R É S U M É

Dans cette Note, nous effectuons une analyse asymptotique uniforme des polynômes de Meixner–Pollaczek $P_n^{(\lambda_n)}(z;\phi)$ avec un paramètre $\lambda_n = (n+\frac{1}{2})A$ lorsque $n\to\infty$, où $A>0$ est une constante. Des développements asymptotiques en termes de fonctions paraboliques cylindriques et de fonctions élémentaires sont obtenus de manière uniforme en z dans deux régions qui recouvrent tout le plan complexe.

Version française abrégée

Dans cette Note, nous effectuons une analyse asymptotique uniforme des polynômes de Meixner–Pollaczek lorsque le degré tend vers l'infini. Ces polynômes ont été découverts par Meixner [8] en 1934, puis étudiés par Pollaczek [10] en 1950. Les polynômes de Meixner–Pollaczek $P_n^{(\lambda)}(x;\phi)$ avec des paramètres $\lambda>0$ et $\phi\in(0,\pi)$ peuvent être définis comme suit à l'aide des fonctions hypergéométriques :

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi}\, {}_2F_1\left(\begin{matrix}-n, \lambda+ix\\ 2\lambda\end{matrix}; 1-e^{-2i\phi}\right).$$

Ils sont orthogonaux sur $\mathbb{R}$ pour la fonction-poids

$$w(x;\lambda,\phi) = \left|\Gamma(\lambda+ix)\right|^2 \exp\left\{(\pi-2\phi)x\right\},$$

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et satisfont la condition d'orthogonalité

$$\int_{-\infty}^{+\infty} P_m^{(\lambda)}(x;\phi)P_n^{(\lambda)}(x;\phi)w(x;\lambda,\phi)\,\mathrm{d}x = \frac{\Gamma(n+2\lambda)}{(2\sin\phi)^{2\lambda}n!}\delta_{mn}.$$

À notre connaissance, il y a peu de travaux sur le comportement asymptotique de ces polynômes ; voir Chen & Ismail [2], Krasovsky [5] pour les zéros extrêmes et la distribution des zéros, et Li & Wong [7] pour leur analyse asymptotique uniforme et des précisions sur le comportement asymptotique des zéros. Dans ces travaux, le paramètre λ est fixé. Dans cette Note, nous analysons le comportement asymptotique uniforme de ces polynômes lorsque le paramètre λ tend vers $+\infty$ lorsque $n \to +\infty$, en considérant le cas où $\lambda = (n+\frac{1}{2})A$ et $A > 0$ est une constante. Seuls les résultats principaux sont présentés ici ; les démonstrations détaillées feront l'objet d'une publication séparée.

L'analyse asymptotique uniforme de polynômes avec des paramètres variables a été effectuée par de nombreux auteurs ; par exemple, par Kuijlaars & McLaughlin [6] pour les polynômes de Laguerre ; par Wong & Zhang [12] pour les polynômes de Jacobi ; et par Baik & Suidan [1] pour les polynômes de Stieltjes–Wigert.

1. MRS numbers and equilibrium measure

Our method is based on the Riemann–Hilbert approach introduced by Deift and Zhou in [3].

Let $\pi_n(z)$ denote the monic polynomials of $P_n^{(\lambda)}(z;\phi)$; i.e.,

$$\pi_n(z) = \frac{n!}{(2\sin\phi)^n}P_n^{(\lambda)}(z;\phi).$$

Let $Y : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{2\times 2}$ be the 2×2 matrix-valued function

$$Y(z) = \begin{pmatrix} \pi_n(z) & C[\pi_n w](z) \\ c_n\pi_{n-1}(z) & c_n C[\pi_{n-1}w](z) \end{pmatrix},$$

where $c_n = -2\pi i(2\sin\phi)^{2(n+\lambda-1)}/[(n-1)!\Gamma(n+2\lambda-1)]$ and $C[f](z)$ is the Cauchy transform of f. From a well-known result of Fokas, Its and Kitaev [4], $Y(z)$ satisfies the following Riemann–Hilbert problem (RHP):

(Y_a) $Y(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$;

(Y_b) for $x \in \mathbb{R}$,

$$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w(x;\lambda,\phi) \\ 0 & 1 \end{pmatrix};$$

(Y_c) for $z \in \mathbb{C}\setminus\mathbb{R}$,

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad \text{as } z \to \infty.$$

Set $\lambda = \lambda_n = \tau A$ with $\tau = n + \frac{1}{2}$ and $A > 0$, and let $w_n(x) = w(\tau x; \tau A, \phi)$. Use the rescaling transform $U(z) = \left(\begin{smallmatrix} \tau^{-n} & 0 \\ 0 & \tau^n \end{smallmatrix}\right)Y(\tau z)$. It is easily shown that $U(z)$ satisfies a RHP which is similar to that for $Y(z)$, but with the jump matrix $\left(\begin{smallmatrix} 1 & w_n(x) \\ 0 & 1 \end{smallmatrix}\right)$.

To obtain the asymptotic behavior of $U(z)$, we first give the equilibrium measure $\mu_n(x)\,\mathrm{d}x$ associated with the weight function $w_n(x)$ which is supported on the interval $[\alpha_n, \beta_n]$, where α_n and β_n are known as the Mhaskar–Rakhmanov–Saff (MRS) numbers determined by

$$\int_{\alpha_n}^{\beta_n} \frac{h_n(s)}{\sqrt{(\beta_n - s)(s-\alpha_n)}}\,\mathrm{d}s = 0 \quad \text{and} \quad \int_{\alpha_n}^{\beta_n} \frac{s h_n(s)}{\sqrt{(\beta_n - s)(s-\alpha_n)}}\,\mathrm{d}s = 2\pi,$$

and $h_n(z) = -\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}z}\log w_n(z) = i[\psi(\tau A - i\tau z) - \psi(\tau A + i\tau z)] - (\pi - 2\phi)$, $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Gamma function. By using the asymptotic expansion of the Psi function $\psi(z)$ as $z \to \infty$ in $|\arg z| < \pi$, we have the following result:

Lemma 1. *The MRS numbers α_n, β_n have the asymptotic expansions*

$$\alpha_n \sim \sum_{k=0}^{\infty}\frac{a_k}{\tau^k} \quad \text{and} \quad \beta_n \sim \sum_{k=0}^{\infty}\frac{b_k}{\tau^k} \quad \text{as } n \to \infty,$$

where the leading coefficients are given by

$$a_0 = \frac{(A+1)\cos\phi - \sqrt{2A+1}}{\sin\phi} \quad \text{and} \quad b_0 = \frac{(A+1)\cos\phi + \sqrt{2A+1}}{\sin\phi},$$

and the higher coefficients a_k, b_k can be determined recursively.

Let $\sigma_n(z) = \sqrt{(z-\alpha_n)(z-\beta_n)}$ for $z \in \mathbb{C} \setminus [\alpha_n, \beta_n]$ such that $\sigma_n(z) \sim z$ as $z \to \infty$, and set

$$G_n(z) = \frac{\sigma_n(z)}{2\pi^2 i} \int_{\alpha_n}^{\beta_n} \frac{h_n(s)}{\sqrt{(\beta_n - s)(s-\alpha_n)}} \frac{1}{s-z} \, \mathrm{d}s.$$

Then, $G_n(z)$ is analytic in $\mathbb{C} \setminus [\alpha_n, \beta_n]$. The equilibrium measure is defined by $\mu_n(x) = \operatorname{Re} G_{n,+}(x)$ for $x \in (\alpha_n, \beta_n)$, where the $+$ sign indicates the limiting value of $\operatorname{Re} G_n(z)$ as z approaches $x \in (\alpha_n, \beta_n)$ from the upper-half plane. Define $g_n(z)$ by $G_n(z) = -\frac{1}{i\pi} g_n'(z)$ for $z \in \mathbb{C} \setminus [\alpha_n, \beta_n]$. It can be shown that

$$g_n(z) = \int_{\alpha_n}^{\beta_n} \log(z-x)\mu_n(x)\,\mathrm{d}x, \quad z \in \mathbb{C} \setminus (-\infty, \beta_n].$$

This function is called the logarithmic potential of $\mu_n(x)$. The following result gives an asymptotic expansion for $\mu_n(x)$ as $n \to \infty$:

Lemma 2. *The equilibrium measure $\mu_n(x)$ has the uniform asymptotic expansion*

$$\mu_n(x) = \frac{1}{2\pi} \log \frac{C(x) + D(x)}{C(x) - D(x)} + \frac{\sqrt{(\beta_n - x)(x-\alpha_n)}}{4\pi\tau} F_n(x),$$

$$F_n(x) \sim \frac{1}{(x+iA)\sigma_n(-iA)} + \frac{1}{(x-iA)\sigma_n(iA)} + \sum_{k=1}^{\infty} \frac{(-1)^k B_{2k}}{k\tau^{2k-1}} \chi_k(x), \quad n \to \infty,$$

for $x \in [\alpha_n, \beta_n]$, where $\tau = n + \frac{1}{2}$, B_{2k} is the $2k$-th Bernoulli number,

$$C(x) = (x-\alpha_n)\sqrt{\beta_n^2 + A^2} - (x-\beta_n)\sqrt{\alpha_n^2 + A^2},$$
$$D(x) = 2\sqrt{(\beta_n - x)(x-\alpha_n)}\, \operatorname{Im} \sqrt{(iA - \beta_n)(-iA - \alpha_n)},$$

and

$$\chi_k(x) = \frac{i}{(2k-1)!} \frac{\mathrm{d}^{2k-1}}{\mathrm{d}s^{2k-1}} \left(\frac{1}{s-x} \frac{1}{\sqrt{(s-\alpha_n)(s-\beta_n)}} \right) \bigg|_{iA}^{-iA}.$$

The square root $\sqrt{(iA - \beta_n)(-iA - \alpha_n)}$ takes the argument in $(-\pi/2, \pi/2)$.

2. An auxiliary function

Let $\nu_n(z)$ be an analytic function in $\mathbb{C} \setminus \{[\alpha_n, \beta_n] \cup (-i\infty, -iA] \cup [iA, i\infty)\}$ satisfying $\nu_{n,\pm}(x) = \pm i\pi\mu_n(x)$ for $x \in (\alpha_n, \beta_n)$. It can be given explicitly by $\nu_n(z) = i\pi G_n(z) + \frac{1}{2} h_n(z)$. Define the auxiliary function $\phi_n(z)$ by

$$\phi_n(z) = \int_{\beta_n}^{z} \nu_n(s)\,\mathrm{d}s, \quad z \in \mathbb{C} \setminus \big\{(-\infty, \beta_n] \cup (-i\infty, -iA] \cup [iA, i\infty)\big\}.$$

The relation between $\phi_n(z)$ and $g_n(z)$ is given by

$$g_n(z) + \phi_n(z) = -\frac{1}{2\tau} \log w_n(z) + \frac{1}{2\tau} \log \frac{\beta_n - \alpha_n}{4} + \frac{1}{2}\ell_n,$$

where ℓ_n is a constant which can be determined by setting $z \to \beta_n$.

For given $0 < c < 1$ and $M > \max\{|\alpha_n|, |\beta_n|\}$, define the rectangle $K = K(c, M) = \{z \in \mathbb{C}\colon\ |\operatorname{Re} z| < M,\ |\operatorname{Im} z| < cA\}$, and let $K_\pm$ denote the upper and lower half of K accordingly. The mapping properties of $\phi_n(z)$ are given below.

Lemma 3. *If $x \in [\beta_n, \infty)$ then $\phi_n(x) \in [0, \infty)$, and when x moves from ∞ to β_n, $\phi_n(x)$ decreases from ∞ to 0. If $x \in [\alpha_n, \beta_n]$ then $\phi_{n,+}(x) \in [-i\pi, 0]$, and when x moves from β_n to α_n, $\phi_{n,+}(x)$ moves from 0 to $-i\pi$ monotonically. If $x \in (-\infty, \alpha_n]$ then $\phi_{n,+}(x) \in [-i\pi, \infty - i\pi)$, and when x moves from α_n to $-\infty$, $\phi_{n,+}(x)$ increases from $-i\pi$ to $\infty - i\pi$. Furthermore, for any $M > \max\{|\alpha_n|, |\beta_n|\}$, there is a constant $c \in (0,1)$ such that $\phi_n(z)$ conformally maps the upper-half rectangle $K_+ = K_+(c, M)$ to a region in $\mathbb{C} \setminus \{z\colon \operatorname{Re} z \geqslant 0, -\pi \leqslant \operatorname{Im} z \leqslant 0\}$.*

With the above preliminaries, we can now follow the standard arguments of the Riemann–Hilbert approach: (1) $U \to T$, the normalization of $U(z)$ at infinity by using logarithmic potential of the equilibrium measure; (2) $T \to S$, the decomposition of the jump matrix and contour deformation; (3) S has a limit, and we denote it by S_∞. Solving the Riemann–Hilbert problem for S_∞, we can get the asymptotic behavior of $U(z)$ outside a neighborhood of $[\alpha_n, \beta_n]$, say, outside the rectangle K. The asymptotic behavior of $U(z)$ outside K is given by

$$U(z) \sim e^{\frac{1}{2}\tau\ell_n\sigma_3}\tilde{V}_{out}(z) w_n(z)^{-\frac{1}{2}\sigma_3}, \quad z \in \mathbb{C} \setminus (K \cup \mathbb{R}),\ n \to \infty,$$

where

$$\tilde{V}_{out}(z) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ \frac{-i(2z-\alpha_n-\beta_n)}{\beta_n-\alpha_n} & -2i \end{pmatrix} b_n(z)^{-\sigma_3}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} e^{-\tau\phi_n(z)\sigma_3},$$

$b_n(z) = [(z-\alpha_n)(z-\beta_n)]^{1/4}/\sqrt{\beta_n - \alpha_n}$ for $z \in \mathbb{C} \setminus (-\infty, \beta_n]$, and σ_3 is the Pauli matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

3. Construction of parametrix

To obtain the asymptotic behavior of $U(z)$ inside K, we need to construct a parametrix $V(z) = \tilde{V}_{in}(z)$ such that

(1) $V_+(x) = V_-(x)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ for $x \in \mathbb{R}$ (jump condition);
(2) it behaves like $\tilde{V}_{out}$ on the boundary of K (matching condition).

The mapping properties of $\phi_n(z)$ invokes us to construct our approximate solution by using the parabolic cylinder function $U(a, z)$. To this end, we introduce the function

$$f(\xi) = \xi\sqrt{\xi^2 - 1} - \log\bigl(\xi + \sqrt{\xi^2 - 1}\bigr), \quad \xi \in \mathbb{C} \setminus (-\infty, 1].$$

This function plays an important role in describing the asymptotic behavior of the parabolic cylinder function $U(-\tau, 2\sqrt{\tau}\xi)$ as $\tau \to \infty$; see [9]. The function $f(\xi)$ maps the upper-half plane $\mathbb{C}_+$ conformally onto the region $\mathbb{C} \setminus \{z\colon \operatorname{Re} z \geqslant 0, -\pi \leqslant \operatorname{Im} z \leqslant 0\}$. By comparing it with $\phi_n(z)$, we have a one-to-one correspondence between $\xi \leftrightarrow z$ defined by

$$f\bigl(\xi(z)\bigr) = \phi_n(z), \quad \text{or} \quad \xi(z) = \bigl(f^{-1} \circ \phi_n\bigr)(z), \quad \text{for } z \in K.$$

With this correspondence, and on account of the connection formula [9, p. 133] for the parabolic cylinder function $U(a, z)$ and the uniform asymptotic expansions [9, pp. 140–143] of $U(-\tau, 2\sqrt{\tau}\xi)$ and $U(\tau, \mp 2i\sqrt{\tau}\xi)$ as $\tau \to \infty$, we now construct the parametrix

$$\begin{aligned}\tilde{V}_{in}(z) = {} & \frac{1}{\sqrt{2}} e^{\frac{\tau}{2}} \tau^{-\frac{n}{2}} \begin{pmatrix} 1 & 0 \\ \frac{-i(2z-\alpha_n-\beta_n)}{\beta_n-\alpha_n} & -2i \end{pmatrix} \left(\frac{(\xi^2-1)^{\frac{1}{4}}}{b_n(z)}\right)^{\sigma_3} \\ & \times \begin{pmatrix} U(-\tau, 2\sqrt{\tau}\xi) & \pm\frac{\Gamma(n+1)}{\sqrt{2\pi}} e^{\mp i\pi n/2} U(\tau, \mp 2i\sqrt{\tau}\xi) \\ \frac{1}{\sqrt{\tau}} U'(-\tau, 2\sqrt{\tau}\xi) & \pm\frac{\Gamma(n+1)}{\sqrt{2\pi\tau}} e^{\mp i\pi(n+1)/2} U'(\tau, \mp 2i\sqrt{\tau}\xi) \end{pmatrix}\end{aligned}$$

for $z \in K_\pm$. Let

$$\tilde{U}(z) = \begin{cases} e^{\frac{1}{2}\tau\ell_n\sigma_3}\tilde{V}_{in}(z) w_n(z)^{-\frac{1}{2}\sigma_3}, & z \in K \setminus \mathbb{R}, \\ e^{\frac{1}{2}\tau\ell_n\sigma_3}\tilde{V}_{out}(z) w_n(z)^{-\frac{1}{2}\sigma_3}, & z \in \mathbb{C} \setminus (K \cup \mathbb{R}). \end{cases}$$

Formally, we have $U(z) \sim \tilde{U}(z)$. To give a rigorous proof and to obtain the asymptotic expansion of $U(z)$, we set the matrix

$$R(z) \equiv e^{-\frac{\tau}{2}\ell_n\sigma_3} U(z)\tilde{U}(z)^{-1} e^{\frac{\tau}{2}\ell_n\sigma_3}.$$

It is easily verified that $R(z)$ is a solution of the following RHP:

(R_a) $R(z)$ is analytic in $z \in \mathbb{C} \setminus \Sigma$, where $\Sigma = \partial K \cup (-\infty, -M] \cup [M, \infty)$;
(R_b) $R(z)$ satisfies the jump condition $R_+(z) = R_-(z) J_R(z)$ for $z \in \Sigma$, where the jump matrix $J_R(z)$ comes from the jump between $\tilde{V}_{out}(z)$ and $\tilde{V}_{in}(z)$ for $z \in \partial K$;
(R_c) $R(z) = I + O(1/z)$ for $z \to \infty$ in $\mathbb{C} \setminus \Sigma$.

4. Main result

Using the uniform asymptotic expansion of the parabolic cylinder function, we can obtain an asymptotic expansion for the jump matrix $J_R(z)$:

$$J_R(z) \sim I + \sum_{k=1}^{\infty} \frac{J_k(z)}{\tau^k}, \quad z \in \Sigma,\ n \to \infty.$$

The coefficients $J_k(z)$ all satisfy $J_k(z) = O(1)$ as $z \to \infty$. By the asymptotic analysis for the Riemann–Hilbert problem [11], we can establish rigorously that $R(z)$ has the uniform asymptotic expansion

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{R_k(z)}{\tau^k}, \quad n \to \infty,$$

on $\mathbb{C} \setminus \Sigma$. Taking the $(1,1)$-entry in the matrix $U(z)$, we have

Theorem 4. *With the above notations, we have*

$$\pi_n(\tau z) = \frac{1}{\sqrt{2}} e^{\frac{\tau}{2}(\ell_n+1)} \tau^{\frac{n}{2}} w_n(z)^{-\frac{1}{2}} \big[U\big(-\tau, 2\sqrt{\tau}\xi(z)\big)A(z,n) + U'\big(-\tau, 2\sqrt{\tau}\xi(z)\big)B(z,n)\big]$$

for $z \in K$, where $A(z,n)$ and $B(z,n)$ are analytic functions of z, and as $n \to \infty$

$$A(z,n) \sim \frac{(\xi^2-1)^{\frac{1}{4}}}{b_n(z)} \left[1 + \sum_{k=1}^{\infty} \frac{A_k(z)}{\tau^k}\right], \qquad B(z,n) \sim \frac{b_n(z)}{(\xi^2-1)^{\frac{1}{4}}} \sum_{k=1}^{\infty} \frac{B_k(z)}{\tau^{k+\frac{1}{2}}}$$

uniformly in K. The coefficients $A_k(z)$ and $B_k(z)$ are all analytic functions in K.

When $z \in \mathbb{C} \setminus K$, we also have the asymptotic expansion

$$\pi_n(\tau z) \sim \frac{\tau^n}{\sqrt{\beta_n - \alpha_n}} e^{\tau g_n(z)} b_n(z)^{-1} \left[1 + \sum_{k=1}^{\infty} \frac{C_k(z)}{\tau^k}\right] \quad \text{as } n \to \infty$$

which holds uniformly in $\mathbb{C} \setminus K$. The coefficients $C_k(z)$ are analytic functions in $\mathbb{C} \setminus K$. The behavior of $\pi_n(\tau z)$ on the boundary of K can be obtained by taking the limit from either inside or outside of K.

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