



## Complex Analysis

## Special polyhedra for Reinhardt domains

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## ABSTRACT

We show that every bounded hyperconvex Reinhardt domain can be approximated by special polynomial polyhedra defined by homogeneous polynomial mappings. This is achieved by means of approximation of the pluricomplex Green function of the domain with pole at the origin.

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## R É S U M É

Nous montrons que tout domaine de Reinhardt borné et hyperconvexe est approché extérieurement par des polyèdres polynomiaux spéciaux définis par des applications polynomiales homogènes. Ceci se fait à l'aide d'une certaine approximation de la fonction de Green pluricomplexe du domaine avec pôle à l'origine.

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## 1. Introduction and statement of results

We consider here the following problem: *Given a polynomially convex compact set  $K \subset \mathbb{C}^n$  and its open neighborhood  $U$ , find a polynomial mapping  $P = (P_1, \dots, P_n)$  with  $P^{-1}(0) \subseteq K$  and such that*

$$K \subset \{z \in \mathbb{C}^n : |P_k(z)| < 1, 1 \leq k \leq n\} \Subset U.$$

When  $n = 1$ , its solution (D. Hilbert, 1897) is known as the Hilbert Lemniscate theorem; a proof can be found, e.g., in [6]. For  $n = 2$ , the problem was solved in [3] in the case of a circled set  $K$  (that is,  $\zeta K \subseteq K$  for any  $\zeta$  from the closed unit disk), and the components of the mapping  $P$  can be chosen as homogeneous polynomials of equal degree with unique common zero at the origin.

For arbitrary  $n \geq 2$ , it was shown in [4,5] that weaker approximations are possible. Namely, either the mapping  $P$  is allowed to have a small part of its zero set outside  $K$  (for the general case of regular sets  $K$ ), or with the number of components of the homogeneous mapping  $P$  increased to  $n + 1$  (for the case of circled  $K$ ). Even the approximation of the closed unit ball by the sublevel sets of  $n$  homogeneous polynomials with  $P^{-1}(0) = \{0\}$  was stated in [5] as an open problem.

In this note, we solve the approximation problem by  $n$  homogeneous polynomials for any multicircled (Reinhardt) polynomially convex compact set  $K$ . For pluripotential theory on multicircled sets and functions, see, for example, [1,7–9].

The above results in [3–5] make use of approximation of the pluricomplex Green function for a compact set with pole at infinity by logarithms of moduli of equidimensional polynomial mappings. Our approach is based on approximation of

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pluricomplex Green functions for bounded Reinhardt domains with pole at the origin. Note however that the Green function with logarithmic pole at 0 for a circled domain  $D$  can be extended by homogeneity  $G(cz) = G(z) + \log |c|$  to the whole space, and the extension coincides on  $\mathbb{C}^n \setminus D$  with the Green function for  $\bar{D}$  with pole at infinity.

The main result is as follows:

**Theorem 1.** *Let  $G(z)$  be the pluricomplex Green function of a bounded, polynomially convex Reinhardt domain  $D \subset \mathbb{C}^n$ , with pole at 0. Then there exists a sequence of mappings  $P^{(s)} : \mathbb{C}^n \rightarrow \mathbb{C}^n$  whose components  $P_k^{(s)}$  are homogeneous polynomials of the same degree  $q_s$ , with the only common zero at 0, such that the sequence of functions  $v_s = q_s^{-1} \max_k \log |P_k^{(s)}|$  converges to  $G$  uniformly on  $D \setminus \{0\}$ , and the functions  $v_s^+ = \max\{v_s, 0\}$  converge to the Green function for  $\bar{D}$  with pole at infinity, uniformly on  $\mathbb{C}^n$ .*

As a consequence, we get an approximation of polynomially convex Reinhardt domains by special polyhedra defined by homogeneous polynomial mappings from  $\mathbb{C}^n$  to  $\mathbb{C}^n$ , which solves, in particular, the problem posed in [5, pp. 366–367].

**Corollary 2.** *For any bounded, polynomially convex Reinhardt domain  $D \subset \mathbb{C}^n$  and every  $\epsilon > 0$ , there exist  $n$  homogeneous polynomials  $p_k$  of the same degree, with the only common zero at 0, such that*

$$\bar{D} \subset \{z: |p_k(z)| < 1, 1 \leq k \leq n\} \subset (1 + \epsilon)D. \tag{1}$$

Finally, since any polynomially convex multicircled compact set is the intersection of domains satisfying the conditions of Corollary 2, we deduce

**Corollary 3.** *For any polynomially convex Reinhardt compact set  $K \subset \mathbb{C}^n$  and every open neighborhood  $U$  of  $K$ , there exist  $n$  homogeneous polynomials  $p_k$  of the same degree, with the only common zero at 0, such that  $K \subset \{z \in \mathbb{C}^n: |p_k(z)| < 1, 1 \leq k \leq n\} \Subset U$ .*

**2. Notation and preliminary results**

Let  $D$  be a bounded hyperconvex domain. By  $G_{a,D}$  we denote the pluricomplex Green function of  $D$  with pole at  $a \in D$ , that is, the upper envelope of negative plurisubharmonic functions  $u$  in  $D$  such that  $u(z) \leq \log |z - a| + O(1)$  as  $z \rightarrow a$ . The function  $G$  is plurisubharmonic in  $D$ , continuous on  $\bar{D}$ , maximal on  $D \setminus \{a\}$ , and  $G(z) = \log |z - a| + O(1)$  near  $a$ .

From now on, we specify  $D$  to be a bounded, logarithmically convex and complete (which amounts to being polynomially convex) Reinhardt ( $n$ -circled) domain in  $\mathbb{C}^n$ , and denote  $G(z) = G_{0,D}(z)$ . Since the domain  $D$  is  $n$ -circled, so is the function  $G$ .

Given  $z \in \mathbb{C}^n$  and  $\theta \in \mathbb{R}_+^n$ , denote  $S(\theta, z) = \sum_k \theta_k \log |z_k|$  and let  $h(\theta) = \sup\{S(\theta, z): z \in D\}$  be the characteristic function of the domain  $D$  (the support function of the logarithmic image  $\log |D|$  of  $D$ ).

**Lemma 1.** (Cf., e.g., [9, Proposition 1.4.3], [1, Lemma 4].) *The Green function  $G$  of a bounded polynomially convex Reinhardt domain  $D \subset \mathbb{C}^n$  with pole at 0 has the representation*

$$G(z) = \sup\{S(\theta, z) - h(\theta): \theta \in \Sigma\}, \tag{2}$$

where  $\Sigma = \{\theta \in \mathbb{R}_+^n: \sum_k \theta_k = 1\}$ .

**Proof.** Denote the right hand side of (2) by  $R(z)$ . As is easy to see, it is plurisubharmonic in  $D$ , equal to zero on  $\partial D$ , equivalent to  $\log |z|$  near 0, and, since  $R(cz) = R(z) + \log |c|$ , it is maximal on  $D \setminus \{0\}$ , so  $R(z) \equiv G(z)$  by the Green function uniqueness property.  $\square$

**Lemma 2.** *For any  $\epsilon > 0$  and  $t < 0$ , there exist finitely many monomials  $g_1, \dots, g_m$  of the same degree  $q$ , such that*

$$|G - v| < \epsilon \quad \text{on } D_{t,0} = \{z: t \leq G(z) \leq 0\}, \tag{3}$$

where

$$v(z) = q^{-1} \max\{\log |g_j(z)|: 1 \leq j \leq m\}, \tag{4}$$

and the maximum is attained for at most  $n$  values of the indices  $j$  at any point  $z$  on the level set

$$\Gamma_t(v) = \{z \in D: v(z) = t\}.$$

**Remark.** Lemma 2 can be deduced from a similar result [9, Lemma 6] on relative extremal functions. Note that it was claimed there that, moreover, the maximum is attained for at most  $n$  functions  $g_j$  on a set corresponding in our case to  $\{-1 \leq v(z) \leq 0\}$ . However the proof of the claim has a gap, and what is actually shown there is that the approximating function possesses this property only on finitely many its level surfaces. That is why we just follow the relevant arguments from the proof of [9, Lemma 6].

**Proof.** Representation (2), continuity of  $G$ , and compactness of  $D_{t,0}$  imply the existence of points  $\theta^{(j)} \in \Sigma$ ,  $1 \leq j \leq m$ , such that  $|G - u| < \epsilon/2$  on  $D_{t,0}$ , where

$$u(z) = \max_{1 \leq j \leq m} \{S(a^{(j)}, z) - b^{(j)}\}, \quad a^{(j)} = \theta^{(j)}, \quad b^{(j)} = h(\theta^{(j)}).$$

What we need to do is to approximate the function  $u$  by a similar one with rational coefficients  $\tilde{a}^{(j)}$  and to provide the required condition about  $n$  values. Consider the space  $X \approx \mathbb{R}^{m-p} \times \mathbb{R}^m$  of matrices  $\mathcal{M} = (a_{jk}; b_j) \in \mathbb{R}^{m \times m} \times \mathbb{R}^m$  such that  $a_{jk} = 0$  if  $a_k^{(j)} = 0$  ( $p$  being the number of these equations). For any  $t < 0$ , there exists an algebraic set  $\mathcal{A}_t \subset X$  such that any system

$$\sum_k a_{jk} x_k = b_j + t, \quad j \in J, \quad |J| > n,$$

has no solution for  $(A, b) \in X \setminus \mathcal{A}_t$ . Therefore, one can replace the points  $(a^{(j)}, b^{(j)})$  by  $(\tilde{a}^{(j)}, \tilde{b}^{(j)}) \in X \setminus \mathcal{A}_t$  with rational  $\tilde{a}^{(j)} \in r\Sigma$  for some rational  $r$  close to 1, such that the function

$$v(z) = \max_{1 \leq j \leq m} \{S(\tilde{a}^{(j)}, z) - \tilde{b}^{(j)}\}$$

satisfies (3) and the maximum is attained for at most  $n$  functions at any point of the set  $\Gamma_t(v)$ .

Finally, by choosing  $N \in \mathbb{Z}_+$  so that  $\tilde{a}^{(j)} = N^{-1}k^{(j)}$ ,  $rN \in \mathbb{Z}_+^n$ , and  $k^{(j)} \in \mathbb{Z}_+^n$  for all  $j$ , we get (4) with monomials

$$g_j(z) = e^{-N\tilde{b}^{(j)}} z^{k^{(j)}}$$

of degree  $q = rN$ , which completes the proof.  $\square$

The next point is a construction of precisely  $n$  polynomials approximating  $G$  on  $\Gamma_t(v)$ . To this end, we use a procedure from [9, Lemma 2], see also [2, Theorem 1]. Let  $g_1, \dots, g_m$  be the monomials from Lemma 2. Given  $s \in \mathbb{Z}_+$ , let  $g^{(s)}$  be a polynomial mapping with the components

$$g_k^{(s)} = \left( \sum_{j_1 < \dots < j_k} g_{j_1}^s \dots g_{j_k}^s \right)^{n/k}, \quad k = 1, \dots, n. \tag{5}$$

**Lemma 3.** (See [9, Lemma 2].) *In the above notation, the sequence of functions*

$$v_s = (qsn!)^{-1} \max_{1 \leq k \leq n} \log |g_k^{(s)}| \tag{6}$$

converges to  $v$  uniformly on the level set  $\Gamma_t(v)$ .

### 3. Proofs

**Proof of Theorem 1.** From Lemmas 2 and 3, we derive

$$|G(z) - v_s(z)| < \epsilon, \quad z \in \Gamma_{-1}(v) = \{v(z) = -1\}, \quad s \geq s_0(\epsilon). \tag{7}$$

As follows from Lemma 1, the Green function  $G$  satisfies  $G(cz) = G(z) + \log |c|$  for all  $c \in \mathbb{C}$  such that  $cz \in D$ . The functions  $g_k^{(s)}$  defined by (5) are homogeneous polynomials of degree  $q_s = qsn!$  and thus the function  $v_s$  defined by (6) has the property  $v_s(cz) = v_s(z) + \log |c|$  for all  $c \in \mathbb{C}$ . The homogeneity of both  $G$  and  $v_s$  extends (7) to  $\mathbb{C}^n \setminus \{0\}$  and implies the claimed convergence of the functions  $v_s^+$ .  $\square$

**Proof of Corollary 2.** Take  $\delta = \frac{1}{2} \log(1 + \epsilon)$ . By Theorem 1, there exist  $n$  homogeneous polynomials  $P_k$  of degree  $q$  such that  $|G(z) - q^{-1} \max_k \log |P_k(z)|| < \delta$  for all  $z \in \mathbb{C}^n$ ,  $z \neq 0$ . Then

$$\{z: |P_k|^{1/q} < e^{-\delta}, 1 \leq k \leq n\} \subset \bar{D} \subset \{z: |P_k|^{1/q} < e^\delta, 1 \leq k \leq n\}, \tag{8}$$

which gives (1) with  $p_k(z) = P_k(e^{-\delta}z)$ .  $\square$

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