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Combinatorics

Binding numbers and $[a, b]$-factors excluding a given k-factor ☆

Nombre de liaisons et $[a, b]$-facteur excluant un k-facteur donné

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ABSTRACT

Let G be a graph of order n, and let a, b, k be nonnegative integers with $1 \leqslant a \leqslant b$. An $[a, b]$-factor of G is defined as a spanning subgraph F of G such that $a \leqslant d_F(x) \leqslant b$ for each $x \in V(G)$. If $a = b = k$, then an $[a, b]$-factor is called a k-factor. In this Note, it is proved that if G has a k-factor Q, $n \geqslant \frac{(a+b-1)^2}{b}$, the binding number $bind(G) \geqslant \frac{(a+b-1)(n-1)}{bn-(k+1)(a+b-1)}$, and $|N_G(X)| \geqslant \frac{(a-1)n+(ka+kb-k+1)|X|}{a+b-1}$ for any nonempty independent subset X of $V(G)$, then G has an $[a, b]$-factor F such that $E(F) \cap E(Q) = \emptyset$.

RÉSUMÉ

Soit G un graphe d'ordre n et a, b, k des entiers positifs tels que $1 \leqslant a \leqslant b$. Un $[a, b]$-facteur est défini comme étant un sous-graphe couvrant F de G tel que $a \leqslant d_F(x) \leqslant b$ pour tout $x \in V(G)$. Si $a = b = k$, alors un $[a, b]$-facteur est appelé k-facteur. Dans cette Note on démontre que si G a un k-facteur Q, $n \geqslant \frac{(a+b-1)^2}{b}$, le nombre de liaisons $bind(G) \geqslant \frac{(a+b-1)(n-1)}{bn-(k+1)(a+b-1)}$ et $|N_G(X)| \geqslant \frac{(a-b)n+(ka+kb-k+1)|X|}{a+b-1}$ pour tout sous-ensemble X non vide indépendant de $V(G)$, alors G a un $[a, b]$-facteur F tel que $E(F) \cap E(Q) = \emptyset$.

1. Introduction

The readers are referred to [1] for undefined terms and concepts. All graphs considered are finite and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For each $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. We use $\delta(G)$ to denote the minimum degree of G. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{x \in S} N_G(x)$ and we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. The binding number of G is defined by $bind(G) = \min\{\frac{|N_G(X)|}{|X|}\colon \emptyset \neq X \subset V(G), N_G(X) \neq V(G)\}$.

Let a and b be integers with $1 \leqslant a \leqslant b$. An $[a, b]$-factor of G is defined as a spanning subgraph F of G such that $a \leqslant d_F(x) \leqslant b$ for every vertex x of G. And if $a = b = k$, then an $[a, b]$-factor is called a k-factor.

Many authors have investigated graph factors [2,3,7,9–11]. Li and Tang [5] showed a sufficient condition for a graph to have an $[a, b]$-factor excluding a given k-factor. Let us first introduce a well-known result which provides a binding number condition for the existence of a k-factor in graphs.

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Theorem 1. *(See Katerinis and Woodall [4].) Let $k \geqslant 2$ be an integer and let G be a graph of order $n \geqslant 4k-6$ and binding number $bind(G)$ such that kn is even and $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$. Then G has a k-factor.*

Theorem 1 has been generalized in several directions. Zhou et al. [10] extended this theorem to $[a,b]$-factors.

Theorem 2. *(See Zhou and Jiang [10].) Let G be a graph of order n, and let a and b be two integers such that $1 \leqslant a < b$. If the binding number $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)+2}$ and $n \geqslant \frac{(a+b-1)(a+b-2)}{b}$, then G has an $[a,b]$-factor.*

Motivated by Theorems 1 and 2, we obtain a binding number condition for graphs to have $[a,b]$-factors excluding a given k-factor. The main results will be given in the following section.

2. Main theorems

First, we show a binding number condition for graphs to have $[a,b]$-factors excluding a given k-factor.

Theorem 3. *Let a, b and k be nonnegative integers such that $1 \leqslant a < b$, and let G be a graph of order n with $n \geqslant \frac{(a+b-1)^2}{b}$, and let G have a k-factor Q. If the binding number $bind(G) \geqslant \frac{(a+b-1)(n-1)}{bn-(k+1)(a+b-1)}$, and $|N_G(X)| \geqslant \frac{(a-1)n+(ka+kb-k+1)|X|}{a+b-1}$ for any nonempty independent subset X of $V(G)$, then G has an $[a,b]$-factor F such that $E(F) \cap E(Q) = \emptyset$.*

Using Theorem 3, we get a binding number condition for graphs to have $[a,b]$-factors including a given k-factor.

Theorem 4. *Let a, b and k be nonnegative integers such that $k+1 \leqslant a < b$, and let G be a graph of order n with $n \geqslant \frac{(a+b-2k-1)^2}{b-k}$, and let G have a k-factor Q. If the binding number $bind(G) \geqslant \frac{(a+b-2k-1)(n-1)}{(b-k)n-(k+1)(a+b-2k-1)}$, and $|N_G(X)| \geqslant \frac{(a-k-1)n+(ka+kb-2k^2-k+1)|X|}{a+b-2k-1}$ for any nonempty independent subset X of $V(G)$, then G has an $[a,b]$-factor F such that $E(Q) \subseteq E(F)$.*

Proof. By the assumption of Theorem 4, G has a k-factor Q. Let $c = a-k$ and $d = b-k$. Then we have $n \geqslant \frac{(c+d-1)^2}{d}$, $bind(G) \geqslant \frac{(c+d-1)(n-1)}{dn-(k+1)(c+d-1)}$, and $|N_G(X)| \geqslant \frac{(c-1)n+(kc+kd-k+1)|X|}{c+d-1}$ for any nonempty independent subset X of $V(G)$. By Theorem 3, G has a $[c,d]$-factor F' such that $E(F') \cap E(Q) = \emptyset$, and G has an $[a,b]$-factor F $(F = E(F') \cup E(Q))$ such that $E(Q) \subseteq E(F)$. This completes the proof of Theorem 4. □

Unfortunately, the author does not know whether the conditions in Theorems 3 and 4 are best possible or not. Hence, I pose the following conjectures:

Conjecture 1. *Let a, b and k be nonnegative integers such that $1 \leqslant a < b$, and let G be a graph of order n with $n \geqslant \frac{(a+b-1)^2}{b}$, and let G have a k-factor Q. If the binding number $bind(G) \geqslant \frac{(a+b-1)(n-1)}{bn-(k+1)(a+b-1)+1}$, and $|N_G(X)| \geqslant \frac{(a-1)n+(ka+kb-k+1)|X|-1}{a+b-1}$ for any nonempty independent subset X of $V(G)$, then G has an $[a,b]$-factor F such that $E(F) \cap E(Q) = \emptyset$.*

Conjecture 2. *Let a, b and k be nonnegative integers such that $k+1 \leqslant a < b$, and let G be a graph of order n with $n \geqslant \frac{(a+b-2k-1)^2}{b-k}$, and let G have a k-factor Q. If the binding number $bind(G) \geqslant \frac{(a+b-2k-1)(n-1)}{(b-k)n-(k+1)(a+b-2k-1)+1}$, and $|N_G(X)| \geqslant \frac{(a-k-1)n+(ka+kb-2k^2-k+1)|X|-1}{a+b-2k-1}$ for any nonempty independent subset X of $V(G)$, then G has an $[a,b]$-factor F such that $E(Q) \subseteq E(F)$.*

3. The proof of Theorem 3

The proof of Theorem 3 relies heavily on the following lemmas. The condition given in Lemma 3.1 is clearly necessary. Thus its essence is to show that the necessary condition is also sufficient.

Lemma 3.1. *(See Lovász [6].) Let G be a graph, and let a and b be two nonnegative integers with $a < b$. Then G has an $[a,b]$-factor if and only if*

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geqslant 0$$

for any disjoint subsets S and T of $V(G)$, where $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

Lemma 3.2. *(See Woodall [8].) Let G be a graph of order n with $bind(G) \geqslant c$. Then $\delta(G) \geqslant n - \frac{n-1}{c}$.*

Proof of Theorem 3. By the assumption of Theorem 3, G has a k-factor Q. Set $H = G - E(Q)$. Then $V(H) = V(G)$. Hence G has a desired factor if and only if H has an $[a,b]$-factor. By way of contradiction, we assume that H has no $[a,b]$-factor. Then, by Lemma 3.1, there exist two disjoint subsets S and T of $V(H) = V(G)$ such that

$$\delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \leqslant -1. \tag{1}$$

We choose such subsets S and T so that $|T|$ is minimum. Obviously, $T \neq \emptyset$ by (1).

Claim 1. *$d_{H-S}(x) \leqslant a - 1$ for each $x \in T$.*

Proof. If $d_{H-S}(x) \geqslant a$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T. The proof of Claim 1 is complete. □

In the following, we define that $h = \min\{d_{H-S}(x):\ x \in T\}$. Using Claim 1, we have $0 \leqslant h \leqslant a - 1$. The following proof splits into two cases by the value of h:

Case 1. $h = 0$.

Claim 2. *$|N_H(X)| \geqslant \frac{(a-1)n+|X|}{a+b-1}$ for any nonempty independent subset X of $V(H)$.*

Proof. Since $H = G - E(Q)$, we have $|N_H(X)| \geqslant |N_G(X)| - k|X|$. By the assumption of Theorem 3, we obtain

$$\left|N_H(X)\right| \geqslant \left|N_G(X)\right| - k|X| \geqslant \frac{(a-1)n + (ka+kb-k+1)|X|}{a+b-1} - k|X| = \frac{(a-1)n+|X|}{a+b-1}.$$

This completes the proof of Claim 2. □

Set $X = \{x:\ x \in T,\ d_{H-S}(x) = 0\}$. Obviously, $X \neq \emptyset$ since $h = 0$ and X is an independent subset of $V(H)$. According to Claim 2, we deduce

$$|S| \geqslant \left|N_H(X)\right| \geqslant \frac{(a-1)n+|X|}{a+b-1}. \tag{2}$$

From (1), (2) and $|S| + |T| \leqslant n$, we obtain

$$\begin{aligned} -1 &\geqslant \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \geqslant b|S| + |T| - |X| - a|T| \geqslant b|S| - (a-1)\big(n - |S|\big) - |X| \\ &= (a+b-1)|S| - (a-1)n - |X| \geqslant (a+b-1)\frac{(a-1)n+|X|}{a+b-1} - (a-1)n - |X| = 0. \end{aligned}$$

That is a contradiction.

Case 2. $1 \leqslant h \leqslant a - 1$.

Claim 3. *$\delta(H) \geqslant \frac{(a-1)n}{a+b-1} + 1$.*

Proof. According to Lemma 3.2 and $bind(G) \geqslant \frac{(a+b-1)(n-1)}{bn-(k+1)(a+b-1)}$, we obtain

$$\delta(G) \geqslant n - \frac{n-1}{\frac{(a+b-1)(n-1)}{bn-(k+1)(a+b-1)}} = \frac{(a-1)n}{a+b-1} + k + 1. \tag{3}$$

Since $H = G - E(Q)$, we have $\delta(H) = \delta(G) - k$. Combining this equality with (3), we infer

$$\delta(H) = \delta(G) - k \geqslant \frac{(a-1)n}{a+b-1} + 1.$$

This completes the proof of Claim 3. □

By the definition of h, there exists a vertex x_1 in T such that $d_{H-S}(x_1) = h$. Thus, we have

$$\delta(H) \leqslant d_H(x_1) \leqslant d_{H-S}(x_1) + |S| = h + |S|. \tag{4}$$

Using (4) and Claim 3, we obtain

$$|S| \geqslant \delta(H) - h \geqslant \frac{(a-1)n}{a+b-1} + 1 - h. \tag{5}$$

In terms of (1), (5) and $|S| + |T| \leqslant n$, we have

$$\begin{aligned} -1 &\geqslant \delta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \geqslant b|S| + h|T| - a|T| \\ &\geqslant b|S| - (a-h)\big(n - |S|\big) = (a+b-h)|S| - (a-h)n \geqslant (a+b-h)\left(\frac{(a-1)n}{a+b-1} + 1 - h\right) - (a-h)n, \end{aligned}$$

that is,

$$-1 \geqslant (a+b-h)\left(\frac{(a-1)n}{a+b-1} + 1 - h\right) - (a-h)n. \tag{6}$$

Let $f(h) = (a+b-h)(\frac{(a-1)n}{a+b-1} + 1 - h) - (a-h)n$. Then by $1 \leqslant h \leqslant a-1$ and $n \geqslant \frac{(a+b-1)^2}{b}$, we obtain

$$f'(h) = 2h + \frac{bn}{a+b-1} - (a+b+1) \geqslant 2 + \frac{bn}{a+b-1} - (a+b+1) = \frac{bn}{a+b-1} - (a+b-1) \geqslant 0.$$

Hence, $f(h)$ attains its minimum value at $h = 1$. Thus, by (6) we have

$$-1 \geqslant f(h) \geqslant f(1) = (a+b-1)\left(\frac{(a-1)n}{a+b-1} + 1 - 1\right) - (a-1)n = 0,$$

which is a contradiction.

Hence, G has a desired factor, that is, G has an $[a,b]$-factor F such that $E(F) \cap E(Q) = \emptyset$. This completes the proof of Theorem 3.

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