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Numerical Analysis

A remark on supercloseness and extrapolation of the quadrilateral Han element for the Stokes equations

Une remarque concernant la super-approximation et l'extrapolation de l'élement fini de Han pour les équations de Stokes

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ABSTRACT

We analyze the supercloseness properties of the nonconforming quadrilateral Han finite element for the Stokes equations. It is shown that the difference between the discrete solution and the natural interpolation of the continuous solution does not have the supercloseness property. Based on this analysis, we propose a modified interpolation operator which allows for such a result. It is then used to obtain a simple third-order extrapolation scheme.

RÉSUMÉ

Nous présentons une analyse de la super-convergence de l'espace d'élément finis de Han pour les équations de Stokes. Il est démontré que la différence entre la solution discrète et l'interpolé naturel de la solution n'est pas de l'ordre supérieur (« supercloseness »). Basé sur notre analyse, nous proposons une modification de l'opérateur d'interpolation qui possède cette propriété. Cela permet la construction d'un schéma d'extrapolation simple qui est de l'ordre trois.

1. Introduction

Nonconforming finite element methods on quadrilateral meshes [1,4] are well-known in computational fluid mechanics, since they are locally conservation and present advantages concerning iterative solvers [5].

Supercloseness results, which state that the difference between the finite element solution and the interpolation of the continuous solution is asymptotically smaller than the interpolation error, provide the theoretical basis for defect correction and extrapolation. Extrapolation of nonconforming finite elements is a well-studied subject; see for example [2,6,3]. However, results on super-convergence of the Han element for the Stokes equations seem to be missing, and we wish to fill this gap with the present short note.

We first show, that, as opposed to the case of the Poisson equation, the difference between the discrete solution and the natural interpolation of the continuous solution does not have the supercloseness property. We then propose a modified

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interpolation operator which yields second-order accuracy in the pressure and gradients of velocities. This result allows us to construct a simple third-order accurate extrapolation scheme.

Let $\Omega \subset \mathbb{R}^2$ be bounded polygonal domain and $f \in L^2(\Omega)^2$. We denote by $\langle\cdot,\cdot\rangle$ the $L^2(\Omega)$-scalar product and by $\|\cdot\|$ the associated norm; for a subdomain $K \subset \Omega$ and a segment $S \subset \Omega$ similar notation are used. Further, $|\cdot|_k$ denotes the semi-norm in $H^k(\Omega)$, $k \geqslant 1$. Let $V = H_0^1(\Omega)^2$, $Q = L_0^2(\Omega)$, and $U = V \times Q$. For $(v,p) \in U$ and $(w,r) \in U$ we define the continuous bilinear form $a : Q \times Q \to \mathbb{R}$ by $a((v,p),(w,r)) := \langle \nabla v, \nabla w\rangle - \langle p, \operatorname{div} w\rangle + \langle \operatorname{div} v, r\rangle$. With $l(w) := \langle f, w\rangle$, the standard weak formulation of the Stokes equations homogeneous Dirichlet boundary conditions reads: Find $(v,p) \in Q$ such that for all $(w,r) \in Q$

$$a\big((v,p),(w,r)\big) = l(w). \tag{1}$$

We consider finite element meshes h composed of rectangles K with length d_{x_i} in x_i-direction, $i = 1,2$. The set of rectangles is denoted by $\mathcal{K}_h$ and the set of edges by $\mathcal{S}_h$. With a fixed choice of unit normal, $[v_h]_S$ denotes the jump over an internal edge S; in case of a boundary edge, we set $n_S = n_\Omega$ and $[v_h]_S = v_h$.

For the pressure approximation we use the space of piece-wise constants,

$$Q_h := \big\{p_h \in L_0^2(\Omega):\ p_h|_K \in \mathcal{P}^0(K) \text{ for all } K \in \mathcal{K}_h\big\},$$

whereas the velocity approximation is sought in the nonconforming space

$$V_h := \left\{v_h \in L^2(\Omega)^2:\ v_h|_K \in \mathcal{Q}(K)^2\ \forall K \in \mathcal{K}_h,\ \int_S [v_h]_S \,\mathrm{d}s = 0\ \forall S \in \mathcal{S}_h\right\}.$$

Here $\mathcal{P}^k(K)$ $(k \in \mathbb{N})$ and $\mathcal{Q}(K)$ denote the set of polynomials of maximal degree k and the vector space engendered by $\{1, x, y, x^2, y^2\}$ on K, respectively. A local basis of V_h is defined on each cell K with edges S_i, $i = 1,\dots,4$ by means of the functionals $\phi_i(v) := \int_{S_i} v$ for $i = 1,\dots,4$ and $\phi_5(v) := \int_K v$. We define the discrete gradient and divergence operators $\nabla_h : V_h \to L^2(\Omega)^{2\times 2}$ and $\operatorname{div}_h : V_h \to L^2(\Omega)$ by $(\nabla_h v_h)|_K := \nabla(v_h|_K)$ and $(\operatorname{div}_h v_h)|_K := \operatorname{div}(v_h|_K)$, respectively. Clearly, $|v_h|_{1,h} := \|\nabla_h v_h\|$ is a norm in V_h.

Let $U_h := V_h \times Q_h$. The discrete bilinear form $a_h : Q_h \times Q_h \to \mathbb{R}$ reads

$$a_h\big((v_h,p_h),(w_h,r_h)\big) := \langle \nabla_h v_h, \nabla_h w_h\rangle - \langle p_h, \operatorname{div}_h w_h\rangle + \langle \operatorname{div}_h v_h, r_h\rangle,$$

and the discrete problem under consideration is: Find $(v_h,p_h) \in U_h$ such that for all $(w_h,r_h) \in U_h$

$$a_h\big((v_h,p_h),(w_h,r_h)\big) = l(w_h). \tag{2}$$

The definition of a_h extends to the space $U \oplus U_h$, which we equip with the norm $|||(v_h,p_h)|||_h := \sqrt{|v_h|_{1,h}^2 + \|p_h\|^2}$. It is well-known that there exists a mesh-independent constant C such that

$$\sup_{(w_h,r_h)\in U_h\setminus\{0\}} \frac{a_h((v_h,p_h),(w_h,r_h))}{|||(w_h,r_h)|||_h} \geqslant C\big|\big|\big|(v_h,p_h)\big|\big|\big|_h \quad \forall (v_h,p_h) \in U_h, \tag{3}$$

since the discrete spaces fulfill the inf-sup condition for the discrete gradient operator. It is well-known that the first-order error estimate $|||(v,p)-(v_h,p_h)||| \leqslant C d_h(|u|_2 + |p|_1)$ holds, where d_h denotes the maximal cell diameter. This error estimate uses standard results for the natural interpolation operator $I_h : U \to U_h$, $I_h = (\Pi_h, J_h)$ with $\Pi_h : V \to V_h$ and $J_h : Q \to Q_h$ defined by

$$\phi_K\big(\Pi_h(v)\big) = \phi_K(v) \quad \forall K \in \mathcal{K}_h, \qquad \phi_S\big(\Pi_h(v)\big) = \phi_S(v) \quad \forall S \in \mathcal{S}_h, \qquad \phi_K\big(J_h(p)\big) = \phi_K(p) \quad \forall K \in \mathcal{K}_h.$$

As we show in the next section, the natural interpolation operator I_h does not yield a supercloseness property. However, our analysis allows us to construct a modification which yields such a result.

2. Supercloseness analysis

First we state a bound of the difference between the discrete solution and the interpolation of the continuous solution in terms of the interpolation error.

Lemma 2.1. *Let $(v,p) \in H^3(\Omega)^2 \times H^2(\Omega)$. There is a mesh-independent constant C such that*

$$C\big|\big|\big|(v_h,p_h) - I_h(v,p)\big|\big|\big|_h \leqslant \sup_{(w_h,r_h)\in U_h\setminus\{0\}} \frac{a_h((v,p) - I_h(v,p),(w_h,r_h)) + \mathcal{C}_h}{|||(w_h,r_h)|||_h}, \tag{4}$$

where the consistency error satisfies

$$\mathcal{C}_h = -\sum_{K\in\mathcal{K}_h}\int_{\partial K}(\nabla v\cdot n_K - p n_K)\cdot w_h, \quad \textit{and} \quad |\mathcal{C}_h| \leqslant C d_h^2\big(|v|_3 + |p|_2\big)\|\nabla_h w_h\|. \tag{5}$$

Proof. Estimate (4) follows from the discrete stability (3), the discrete equation, and integration by parts.

For the proof of (5) let us suppose that the edges S_i of K are numbered in clock-wise sense starting from the left edge. We then have for a single term in the definition of $\mathcal{C}_h$

$$\int_{\partial K}(\nabla v\cdot - p)n_K\cdot w_h = \bigg(\int_{S_1} - \int_{S_3}\bigg)\big(\partial_{x_1}v - (p,0)\big)\cdot w_h\,\mathrm{d}s + \bigg(\int_{S_2} - \int_{S_4}\bigg)\big(\partial_{x_2}v - (0,p)\big)\cdot w_h\,\mathrm{d}s. \tag{6}$$

We use the continuity of w_h and transform each of the terms of (6) to the reference element. For example, for the first term involving the velocities, letting Π_i be the mean values over edges i, the bilinear form

$$B(\hat v,\hat w_h) := \int_{\hat S_1}(I - \hat\Pi_1)\partial_{\hat x}\hat v\cdot\hat w_h\,\mathrm{d}\hat s - \int_{\hat S_3}(I - \hat\Pi_3)\partial_{\hat x}\hat v\cdot\hat w_h\,\mathrm{d}\hat s \leqslant C|\partial_{\hat x}\hat v|_{1,\hat K}|\hat w_h|_{1,\hat K}.$$

It is easy to check that $B(\hat v,\hat w_h) = 0$ for all $\hat v\in\mathcal{P}^1(\hat K)^2$, yielding by Bramble–Hilbert $|B(\hat v,\hat w_h)| \leqslant C|\hat v|_3|\hat w_h|_1$. The scaling argument leads to (5). □

The result of Lemma 2.1 shows that the leading error term is the interpolation error. However, inspection of the natural interpolation operator shows that for any $w_h\in V_h$ and $r_h\in Q_h$, $\langle\nabla_h(v - \Pi_h(v)), \nabla_h w_h\rangle = 0$ and $\langle\mathrm{div}_h(v - \Pi_h(v)), r_h\rangle = 0$, so that the leading error term is the pressure gradient, analyzed by the next lemma denoting $w_h = (w_h^1, w_h^2)$.

Lemma 2.2. *The natural interpolation operator does not provide a supercloseness result since*

$$\big\langle p - J_h(p), \mathrm{div}\,w_h\big\rangle = -\sum_{K\in\mathcal{K}_h}\bigg(\frac{d_{x_1}^2}{3}\int_K\partial_x p\,\partial_{x_1}^2 w_h^1\,\mathrm{d}x + \frac{d_{x_2}^2}{3}\int_K\partial_{x_2}p\,\partial_y^2 w_h^2\,\mathrm{d}x\bigg) + O\big(d_h^2\big)|p|_2|w_h|_{1,h}. \tag{7}$$

Proof. We use transformation to the reference element, on which we define the continuous bilinear form

$$B(\hat p,\hat w_h) := \int_{\hat K}\big(\hat p - \hat J(\hat p)\big)\partial_{\hat x_1}\hat w_h^1\,\mathrm{d}x - \frac{1}{3}\int_{\hat K}\partial_{\hat x_1}\hat p\,\partial_{\hat x_1}^2 w_h^1\,\mathrm{d}x + \int_{\hat K}\big(\hat p - \hat J(\hat p)\big)\partial_{\hat x_2}\hat w_h^2\,\mathrm{d}x - \frac{1}{3}\int_{\hat K}\partial_{\hat x_2}\hat p\,\partial_{\hat x_2}^2 w_h^2\,\mathrm{d}x.$$

It is easy to check that $B(\hat p,\hat w_h) = 0$ if $\hat p\in\mathcal{P}^1(\hat K)$. The Bramble–Hilbert lemma therefore leads to (7). □

The crucial point in (7) is that the second-order derivatives of w_h cannot be bounded without loss of accuracy. We next define a modified interpolation operator which avoids this term. To this end we introduce the linear interpolation operator $\Pi_h^*: U\to V_h$ (depending on p) satisfying the relations

$$\int_S \Pi_h^*(v,p)\,\mathrm{d}s = \int_S v\,\mathrm{d}s \quad \forall S\in\mathcal{S}_h \quad \text{and} \quad \int_K \Pi_h^*(v,p)\,\mathrm{d}s = \int_K\bigg(v + \frac{1}{3}\big(d_{x_1}^2\partial_{x_1}p, d_{x_2}^2\partial_{x_2}p\big)\bigg)\mathrm{d}x \quad \forall K\in\mathcal{K}_h$$

and set $I_h^*(v,p) := (\Pi_h^*(v,p), J_h(p))$. From the preceding analysis we obtain the main result of this section:

Theorem 2.3. *There exists a constant C such that for $(v,p)\in H^3(\Omega)^2\times H^2(\Omega)$, $\|(v,p) - I_h^*(v,p)\|_h \leqslant Cd_h^2(|v|_3 + |p|_2)$.*

3. Extrapolation

In order to obtain a third-order accurate approximation we need to improve the error expansion of Lemma 2.1. This is done in the following lemma, the proof of which is omitted here:

Lemma 3.1. *Under the assumption $(v,p)\in H^4(\Omega)^2\times H^3(\Omega)$ we have*

$$\mathcal{C}_h = \frac{1}{3}\sum_{K\in\mathcal{K}_h}\frac{d_{x_1}^2}{3}\int_K\big(\partial_{x_1}^2\partial_{x_2}p\,w_h^2 - \partial_{x_1}^2\partial_{x_2}^2 v\cdot w_h\big) + \frac{d_{x_2}^2}{3}\int_K\big(\partial_{x_1}\partial_{x_2}^2 p\,w_h^1 - \partial_{x_1}^2\partial_{x_2}^2 v\cdot w_h\big), \tag{8}$$

where $\mathcal{R} = O(d_h^3)(|v|_4 + |p|_3)|w_h|_{1,h}$ and

$$\langle p - J_h(p), \operatorname{div}_h w_h\rangle = -\frac{1}{3}\sum_{K\in\mathcal{K}_h}\int_K \big(d_{x_1}^2\,\partial_{x_1}p\,\partial_{x_1}^2 w_h^1 + d_{x_2}^2\,\partial_{x_2}p\,\partial_{x_2}^2 w_h^2\big)\,\mathrm{d}x + O\big(d_h^3\big)|p|_3|w_h|_{1,h}. \tag{9}$$

The error expansion of Lemma 3.1 together with (4) opens the door to a simple extrapolation scheme based on two meshes h and $h/2$ obtained by regular subdivision. The two discrete pairs of solutions are denoted by (v_h, p_h) and $(v_{h/2}, p_{h/2})$. We need to define interpolation operators Π_h^3 and J_h^2 with the following properties: for all $w_{h/2}\in V_{h/2}$, $v\in H^4(\Omega)^2$, $r_{h/2}\in Q_{h/2}$, and $p\in H^3(\Omega)$

$$\Pi_h^3\Pi_{h/2}^*(v,p) = \Pi_h^3(v),\qquad \big|\Pi_h^3(w_{h/2})\big|_{1,h}\leqslant C|h/2w_{h/2}|_{1,h},\qquad \big|v-\Pi_h^3(v)\big|_{1,h}\leqslant Cd_h^3|v|_4,$$
$$J_h^2\big(J_{h/2}(p)\big) = J_h^2(p),\qquad \big\|J_h^2(r_{h/2})\big\|\leqslant C\|r_{h/2}\|,\qquad \big\|\big(p-J_h^2(p)\big)\big\|\leqslant Cd_h^3|p|_3.$$

We then define

$$v_h^* := \frac{4\Pi_h^3(v_{h/2})-\Pi_{2h}^3(v_h)}{3},\qquad p_h^* := \frac{4J_h^2(p_{h/2})-J_{2h}^2(p_h)}{3}.$$

Theorem 3.2. *Under the assumption $(v,p)\in H^4(\Omega)^2\times H^3(\Omega)$ we have*

$$\big\|v-v_h^*\big\|_h\leqslant Cd_h^3\big(|v|_4+|p|_3\big). \tag{10}$$

A similar result holds for the pressure error, but is omitted here.

Proof. We first note that for arbitrary v^*, since $d_{h/2}=d_h/2$,

$$\begin{aligned}\frac{4\Pi_h^3(v_{h/2})-\Pi_{2h}^3(v_h)}{3} - v &= \frac{4}{3}\big(\Pi_h^3(v_{h/2})-v\big)-\frac{1}{3}\big(\Pi_{2h}^3(v_h)-v\big)\\ &= \frac{4}{3}\big(\Pi_h^3(v_{h/2})+(d_{h/2})^2v^*-v\big)-\frac{1}{3}\big(\Pi_{2h}^3(v_h)+d_h^2v^*-v\big) = \frac{4}{3}F_{h/2}-\frac{1}{3}F_h.\end{aligned}$$

In order to show that F_h (and $F_{h/2}$) are of third order, we let (v^*,p^*) be the solution of

$$a\big((v^*,p^*),(w,r)\big) = \big(f^*,w\big),\qquad f_i^* = -\frac{1}{3}\sum_{K\in\mathcal{K}_h}\frac{d_{x_1}^2+d_{x_2}^2}{d_h^2}\frac{\partial^4 v_i}{\partial x_1^2\partial x_2^2}+\frac{1}{3}\sum_{K\in\mathcal{K}_h}\frac{d_{x_{i+1}}^2}{d_h^2}\frac{\partial^3 p}{\partial x_{i+1}^2\partial x_i}, \tag{11}$$

where the index $i+1$ is modulo 2. Let (v_h^*, p_h^*) be the discrete analog of (11). From regularity we have

$$\big|\!\big|\!\big|\big(v^*,p^*\big)-\big(v_h^*,p_h^*\big)\big|\!\big|\!\big|\leqslant Cd_h\big(\big|v^*\big|_2+\big|p^*\big|_1\big)\leqslant Cd_h\big(|v|_4+|p|_3\big). \tag{12}$$

Let now $\widetilde{v}_h := v_h - \Pi_h^*(v,p) - d_h^2v_h^*$ and $\widetilde{p}_h := p_h - J_h^*(p) - d_h^2p_h^*$. Then we have from (3)

$$\big|\!\big|\!\big|(\widetilde{v}_h,\widetilde{p}_h)\big|\!\big|\!\big|_h\leqslant \sup_{(w_h,r_h)\in U_h\setminus\{0\}}\frac{a_h((v,p)-I_h^*(v,p),(w_h,r_h))+\mathcal{C}_h-h^2a_h((v_h^*,p_h^*),(w_h,r_h))}{|\!|\!|(w_h,r_h)|\!|\!|_h}$$

which implies by the previous analysis $|\!|\!|(\widetilde{v}_h,\widetilde{p}_h)|\!|\!|_h\leqslant Cd_h^3(|v|_4+|p|_3)$, and therefore

$$\|F_h\|_h = \big\|\Pi_h^3(\widetilde{v}_h)+\big(\Pi_h^3(v)-v\big)+d_h^2\big(v^*-\Pi_h^3\big(v_h^*\big)\big)\big\|_h\leqslant Cd_h^3\big(|v|_4+|p|_3\big).\qquad\square \tag{13}$$

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