Effective Cartan–Tanaka connections for \( C^6 \)-smooth strongly pseudoconvex hypersurfaces \( M^3 \subset \mathbb{C}^2 \)

**Connections de Cartan–Tanaka effectives pour les hypersurfaces strictement pseudoconvexes \( M^3 \subset \mathbb{C}^2 \) de classe \( C^6 \)**

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Explicit Cartan–Tanaka curvatures, the vanishing of which characterizes sphericity, are provided in terms of the 6-th order jet of a graphing function for a \( C^6 \) strongly pseudoconvex hypersurface \( M^3 \subset \mathbb{C}^2 \).

**Résumé**

Des courbures de Cartan–Tanaka explicites, dont l’annulation identique caractérise la sphéricité, sont fournies en termes du jet d’ordre 6 d’une fonction graphante pour une hypersurface \( M^3 \subset \mathbb{C}^2 \) de classe \( C^6 \) strictement pseudoconvexe.

**1. Cartan connection, curvature function and second cohomology of Lie algebras**

**Definition 1.1.** (See [12,4].) Let \( G \) be a real Lie group with a closed subgroup \( H \), and let \( g \) and \( h \) be the corresponding real Lie algebras. A Cartan geometry of type \((G, H)\) on a \( C^\infty \) manifold \( M \) is a principal \( H \)-bundle \( \pi : P \rightarrow M \) together with a \( g \)-valued ‘Cartan connection’ 1-form \( \omega : T P \rightarrow g \) satisfying:

- (i) \( \omega_p : T_p P \rightarrow g \) is an isomorphism at every \( p \in P \);
- (ii) if \( R_h(p) := ph \) is the right translation on \( G \) by an \( h \in H \), then \( R^*_h \omega = Ad(h^{-1}) \circ \omega \);
- (iii) \( \omega(H^1) = h \) for every \( h \in h \), where \( H^1|_p := \tfrac{d}{dt}|_0 (R_{\exp(t)}(p)) \) is the left-invariant vector field on \( G \) associated to \( h \).

Since the associated curvature 2-form \( \Omega(X, Y) := d\omega(X, Y) + [\omega(X), \omega(Y)]_g \), with \( X, Y \in \Gamma(T P) \), vanishes if either \( X \) or \( Y \) is vertical [12], \( \Omega \) is fully represented by the curvature function \( \kappa \in C^\infty(P, \Lambda^2(g^*/h^*) \otimes g) \) which sends a point \( p \in P \) to the map \( \kappa(p) : (g/h) \wedge (g/h) \rightarrow g \) defined by:

\[
(\chi \mod h) \wedge (\chi' \mod h) \mapsto -\Omega_p(\omega^{-1}_p(\chi'), \omega^{-1}_p(\chi')) = -[\chi, \chi']_g + \omega_p([\widehat{\chi'}, \widehat{\chi''}]),
\]

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where $\tilde{X} := \omega^{-1}(x)$ is the constant field on $\mathcal{P}$ associated to an $x \in \mathfrak{g}$. Denote then $r := \text{dim}_\mathbb{R} \mathfrak{g}$, $n := \text{dim}_\mathbb{R}(\mathfrak{g}/\mathfrak{h})$ whence $n - r = \text{dim}_\mathbb{R} \mathfrak{h}$ and suppose $r \geq 2$, $n > 1$, $n - r > 1$ so that $\mathfrak{g}$, $\mathfrak{g}/\mathfrak{h}$ and $\mathfrak{h}$ all are nonzero. Picking an adapted basis $(x_k)_{1 \leq k \leq r}$ with $\mathfrak{g} = \text{Span}_\mathbb{R}(x_1, \ldots, x_r, x_{r+1}, \ldots, x_n)$ and $\mathfrak{h} = \text{Span}_\mathbb{R}(x_{r+1}, \ldots, x_n)$, one may expand the curvature function by means of $\mathbb{R}$-valued components:

$$\kappa(p) = \sum_{1 \leq i < j \leq n} \sum_{k=1}^r \kappa_{i,j,k}^k(p) x_i^k \wedge x_j^k \otimes x_k.$$

**Lemma 1.2.** (See [2].) For any field $Y^\perp = \frac{d}{dt}|_{t=0}\exp(ty)$ on $\mathcal{P}$ associated to an arbitrary $y \in \mathfrak{h}$, one has:

$$(Y^\perp \kappa)(p)(x', x'') = -[y, \kappa(p)(x', x'')]_{\mathfrak{g}} + \kappa(p)([y, x']_{\mathfrak{g}}, x'') + \kappa(p)(x', [y, x'']_{\mathfrak{g}}).$$

For any $k \in \mathbb{N}$, consider $k$-cochains $\tilde{\mathfrak{g}}^k := \Lambda^k(\mathfrak{g}^*/\mathfrak{h}^*) \otimes \mathfrak{g}$ and define the differential $\partial^k: \tilde{\mathfrak{g}}^k \to \tilde{\mathfrak{g}}^{k+1}$ by:

$$(\partial^k \phi)(z_0, z_1, \ldots, z_k) := \sum_{i=0}^k (-1)^i [z_i, \phi(z_0, \ldots, \hat{z}_i, \ldots, z_k)]_{\mathfrak{g}} + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \phi([z_i, z_j]_{\mathfrak{g}}, z_0, \ldots, \hat{z}_i, \ldots, \hat{z}_j, \ldots, z_k).$$

Especially for $k = 2$, the cohomologies $\mathcal{H}^2 = \ker(\partial^2)/\text{im}(\partial^{k-1})$ encode deformations of Lie algebras and are useful for Cartan connections [13,4,6]. cf. [1] for an algorithm using Gröbner bases.

**Lemma 1.3.** (Bianchi identity [4,6,2].) For any three $x', x'', x''' \in \mathfrak{g}$, one has at every point $p \in \mathcal{P}$:

$$0 = (\partial^2 \kappa)(p)(x', x', x'') + \sum_{\text{cycl}} \kappa(p)(\kappa(p)(x', x'), x''') + \sum_{\text{cycl}} (\tilde{X}'(\kappa))(p)(x'', x''').$$

When $\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ is graded as in Tanaka's theory, with $[\mathfrak{g}_{1\lambda_1}, \mathfrak{g}_{1\lambda_2}]_\mathfrak{g} \subset \mathfrak{g}_{1\lambda_1+\lambda_2}$ for any $\lambda_1, \lambda_2 \in \mathbb{Z}$ and when $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$, cochains enjoy a natural grading, the second cohomology is graded too: $\mathcal{H}^2 = \bigoplus_{h \in \mathbb{Z}} \mathcal{H}^2_{[h]}$, and the graded Bianchi identities [6,2];

$$\partial^2_{[h]}(\kappa_{[h]})(x', x'', x''') = -\sum_{\text{cycl}} \sum_{h' = 1}^{h-1} (\kappa_{[h-h']}([\kappa_{[h']}([x', x'']), x'''])) - \sum_{\text{cycl}} (\tilde{X}'(\kappa_{[h+|x'|]}))(x'', x''')$$

show that the lowest order nonvanishing curvature must be $\partial$-closed, and more generally, any homogeneous curvature component is determined by the lower components up to a $\partial$-closed component.

### 2. Geometry-preserving deformations of the Heisenberg sphere $\mathbb{H}^3 \subset \mathbb{C}^2$

Let now $M^3 \subset \mathbb{C}^2$ be a local strongly pseudoconvex $\mathfrak{g}^6$-smooth real 3-dimensional hypersurface, represented in coordinates $(z, w) = (x + iy, u + iv)$ as the graph (for background, see [7–11]):

$$v = \varphi(x, y, u) = x^2 + y^2 + O(3)$$

of a certain real-valued $\mathfrak{g}^6$ function $\varphi$ defined in a neighborhood of the origin in $\mathbb{R}^3$. Its complex tangent bundle $T^c M = \text{Re} T \mathbb{R}^{0,1} M$ is generated by the two vector fields:

$$H_1 := \frac{\partial}{\partial x} + \left(\frac{\varphi_y - \varphi_x \varphi_u}{1 + \varphi_u^2}\right) \frac{\partial}{\partial u} \quad \text{and} \quad H_2 := \frac{\partial}{\partial y} + \left(-\varphi_x + \varphi_y \varphi_u \varphi_u + \varphi_x \varphi_y \varphi_u - \varphi_y^2 \varphi_u\right) \frac{\partial}{\partial u}.$$}

which make a frame joint with the third, Levi form-type Lie-bracket:

$$T := \frac{1}{4} [H_1, H_2] = \left(\frac{1}{4 (1 + \varphi_u^2)} \left(-\varphi_{xx} - \varphi_{yy} - 2 \varphi_y \varphi_{xx} - \varphi_x^2 \varphi_{uu} + 2 \varphi_x \varphi_y \varphi_u - \varphi_y^2 \varphi_u \varphi_{uu}ight)ight) \frac{\partial}{\partial u}.$$}

Such $M^3$’s are geometry-preserving deformations of the Heisenberg sphere $\mathbb{H}^3$: $v = x^2 + y^2$. It is known that the Lie algebra $\mathfrak{hol}(\mathbb{H}^3) := \{X = Z(z, w) \frac{\partial}{\partial z} + W(z, w) \frac{\partial}{\partial w}, \ X + \bar{X} \text{ tangent to } \mathbb{H}^3\}$ of infinitesimal CR automorphisms of the Heisenberg sphere $\mathbb{H}^3$ in $\mathbb{C}^2$ is 8-dimensional and generated by:
\[ T := \partial_w, \quad H_1 := \partial_z + 2iz\partial_w, \quad H_2 := i\partial_z + 2z\partial_w, \quad D := z\partial_z + 2w\partial_w, \quad R := iz\partial_z, \]
\[ l_1 := (\partial_z + 2iz\partial_w) + 2izw\partial_w, \quad l_2 := (i\partial_z + 2z\partial_w) + 2zw, \quad J := zw\partial_z + w^2\partial_w. \]

Recently, Ezhev, McLaughlin and Schmalz published in the Notices of the AMS an expository article [6] in which they reconstruct — within Tanaka’s framework and assuming that \( M \) is \( \mathcal{C}^0 = \) Cartan’s connection [5] valued the eight-dimensional abstract real Lie algebra:

\[ \mathfrak{g} := \mathfrak{h} \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_4 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \quad (\text{with } \mathfrak{h} := \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3) \]

spanned by some eight abstract commuting tables as \( T, \ldots, J \) (cf. also [3] for a similar approach).

A natural Tanaka grading is: \( \mathfrak{g} = \mathfrak{g}_2 = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2; \quad \mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{h}_1; \quad \mathfrak{g}_1 = \mathfrak{h}_1 \oplus \mathfrak{h}_2; \quad \mathfrak{g}_2 = \mathfrak{h}_2 \; \text{by performing the above} \]

\[ \mathcal{A} \quad \text{of an initial frame for } TM \quad \text{which is explicit in terms of the graphing function } \varphi(x, y, u), \quad \text{we deviate from} \]

the initial normalization made in [6] (with a more geometric-minded approach), since our computational objective is to provide a Cartan–Tanaka connection all elements of which are completely effective in terms of \( \varphi(x, y, u) \) — assuming only \( \mathcal{C}^6 \)-smoothness of \( M \).

Call \( \gamma \) the numerator of \( T = \frac{1}{4}[H_1, H_2] = \frac{1}{4} \gamma \, \omega'' \), allow the two notational coincidences: \( x_1 \equiv x, \; x_2 \equiv y \); introduce the two length-three brackets:

\[ [H_1, T] = \frac{1}{4}[H_1, [H_1, H_2]] =: \Phi T \quad (i = 1, 2), \]

which are both multiples of \( T \) by means of two functions \( \Phi_i := \frac{A_i}{\sqrt{\gamma}} \); lastly, introduce furthermore the \( H_k \)-iterated derivatives of the functions \( \Phi_i \) up to order 3, where \( i, k_1, k_2, k_3 = 1, 2 \):

\[ H_{k_1}(\Phi_i) = \frac{A_{i,k_1}}{\Delta^2 \gamma^2}, \quad H_{k_2}(H_{k_1}(\Phi_i)) = \frac{A_{i,k_1,k_2}}{\Delta^3 \gamma^3}, \quad H_{k_3}(H_{k_2}(H_{k_1}(\Phi_i))) = \frac{A_{i,k_1,k_2,k_3}}{\Delta^4 \gamma^4}. \]

Proposition 2.1. (See [2].) All the numerators appearing above are explicitly given by:

\[ A_1 := \Delta^2 \gamma x_k + \Delta(-2\Delta \gamma \gamma + A_1 \gamma u - \gamma A_1 u) - \Delta(\gamma \gamma \Delta u). \]

\[ A_{i,k_1} := \Delta^2(\gamma A_{i,x_k} - \gamma x_k A_i) + \Delta(-2\Delta x_k \gamma A_i + \gamma A_{k_1} A_{i,u} - \gamma A_k A_i) - 2\Delta u \gamma A_k A_i. \]

\[ A_{i,k_1,k_2} := \Delta^2(\gamma A_{i,k_1,k_2} - 2\gamma x_k A_{i,k_1}) + \Delta(-3\Delta x_k \gamma A_{i,k_1} + \gamma A_{k_2} A_{i,k_1,u} - 2\gamma A_{k_2} A_{i,k_1}) - 3\Delta u \gamma A_{k_2} A_{i,k_1}. \]

\[ A_{i,k_1,k_2,k_3} = \Delta^2(\gamma A_{i,k_1,k_2,k_3} - \gamma x_k A_{i,k_1,k_2}) + \Delta(-3\Delta x_k \gamma A_{i,k_1,k_2} + 2\gamma A_{k_3} A_{i,k_1,k_2,u} - 3\gamma A_{k_3} A_{i,k_1,k_2}). \]

Furthermore, these iterated derivatives identically satisfy \( H_2(\Phi_1) = H_1(\Phi_2) \) and:

\[ 0 = -H_1(H_2(H_1(\Phi_2))) + 2H_2(H_1(H_1(\Phi_2))) - H_2(H_2(H_1(\Phi_1))) - H_2(H_2(H_1(\Phi_1))) - \Phi_2 H_1(H_2(\Phi_1)) + \Phi_2 H_2(H_1(\Phi_1)). \]

\[ 0 = -H_2(H_1(H_2(\Phi_2))) + 2H_1(H_2(H_1(\Phi_2))) - H_1(H_2(H_2(\Phi_2))) - \Phi_1 H_2(H_1(\Phi_2)) + \Phi_1 H_1(H_2(\Phi_2)). \]

\[ 0 = -H_1(H_1(H_2(\Phi_2))) + 2H_2(H_1(H_2(\Phi_1))) - H_2(H_1(H_2(\Phi_1))) + \Phi_1 H_1(H_2(\Phi_1)) - \Phi_1 H_2(H_1(\Phi_1)). \]

\[ 0 = -H_2(H_2(H_1(\Phi_2))) + 2H_2(H_2(H_1(\Phi_2))) - H_1(H_2(H_2(\Phi_2))) + \Phi_2 H_2(H_1(\Phi_2)) - \Phi_2 H_1(H_2(\Phi_2)). \]

3. Explicit Cartan–Tanaka connection

Theorem 3.1. (See [2].) Associated to such an \( M^3 \subset \mathbb{C}^2 \), there is a unique \( \mathfrak{g} \)-valued Cartan connection which is normal and regular in the sense of Tanaka. Its curvature function reduces to:

\[ \kappa(p) = \kappa_{i_1}^{i_1}(p) h_1^i \wedge t^i \otimes i_1 + \kappa_{i_2}^{i_1}(p) h_1^i \wedge t^i \otimes i_2 + \kappa_{i_3}^{i_1}(p) h_1^i \wedge t^i \otimes j_1 \]

\[ + \kappa_{i_4}^{i_1}(p) h_1^i \wedge t^i \otimes i_2 + \kappa_{i_5}^{i_1}(p) h_1^i \wedge t^i \otimes j_2 + \kappa_{i_6}^{i_1}(p) h_1^i \wedge t^i \otimes j_3. \]

where the two main curvature coefficients, having homogeneity 4, are of the form:

\[ \kappa_{i_1}^{i_1}(p) = -\Delta_1 c^4 - 2\Delta c^3 d - 2\Delta d^3 + \Delta d^4 \quad \text{and} \quad \kappa_{i_2}^{i_1}(p) = -\Delta_1 c^4 + 2\Delta_1 c^3 d + 2\Delta_1 c d^3 + \Delta_4 d^4, \]

in which the two functions \( \Delta_1 \) and \( \Delta_4 \) of only the three variables \( (x, y, u) \) are explicitly given by:
\[ \Delta_1 = \frac{1}{384} \left[ H_1 (H_1 (H_1 (\Phi_1))) - H_2 (H_2 (\Phi_2)) + 11 H_1 (H_2 (\Phi_1)) - 11 H_2 (H_1 (\Phi_1)) + 6 \Phi_2 H_2 (H_1 (\Phi_1)) - 6 \Phi_1 H_1 (H_2 (\Phi_2)) - 3 \Phi_2 H_1 (H_1 (\Phi_2)) + 3 \Phi_1 H_2 (H_2 (\Phi_1)) - 3 \Phi_1 H_1 (H_1 (\Phi_2)) + 3 \Phi_2 H_2 (H_2 (\Phi_2)) - 2 \Phi_1 H_1 (\Phi_1) + 2 \Phi_2 H_2 (\Phi_2) - 2 (\Phi_2)^2 H_1 (\Phi_1) + 2 (\Phi_1)^2 H_2 (\Phi_2) - 2 (\Phi_2)^2 H_2 (\Phi_2) + 2 (\Phi_1)^2 H_1 (\Phi_1) \right]. \]

\[ \Delta_4 = \frac{1}{384} \left[ -3 H_2 (H_1 (H_2 (\Phi_2))) - 3 H_1 (H_2 (H_1 (\Phi_1))) + 5 H_1 (H_2 (\Phi_2)) + 5 H_2 (H_1 (\Phi_1)) + 4 \Phi_1 H_1 (H_1 (\Phi_2)) + 4 \Phi_2 H_2 (H_1 (\Phi_2)) - 3 \Phi_2 H_1 (H_1 (\Phi_1)) - 3 \Phi_1 H_2 (H_2 (\Phi_2)) - 7 \Phi_2 H_1 (H_2 (\Phi_2)) - 7 \Phi_1 H_2 (H_1 (\Phi_1)) - 2 H_1 (\Phi_1) H_2 (\Phi_2) - 2 H_2 (\Phi_2) H_1 (\Phi_1) + 4 \Phi_1 \Phi_2 H_1 (\Phi_1) + 4 \Phi_1 \Phi_2 H_2 (\Phi_2) \right], \]

and where the remaining four secondary curvature coefficients are given by:

\[ k_{i_1}^{h_2, t} = k_{i_2}^{h_2, t}, \quad k_{j_1}^{h_2, t} = -k_{i_1}^{h_2, t}, \quad k_{j_1}^{h_2, t} = \hat{H}_1 (k_{i_2}^{h_2, t}) - \hat{H}_2 (k_{i_2}^{h_2, t}), \quad k_{j_1}^{h_2, t} = -\hat{H}_1 (k_{i_1}^{h_2, t}) + \hat{H}_2 (k_{i_1}^{h_2, t}). \]

**Corollary 3.2.** A \( \varphi^{\omega} \)-smooth strongly pseudoconvex local hypersurface \( M^3 \subset \mathbb{C}^2 \) is biholomorphic to \( \mathbb{H}^3 \), namely is spherical, if and only if \( 0 \equiv \Delta_1 \equiv \Delta_4 \), identically as functions of \((x, y, u)\).

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**References**


