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On the growth of Betti numbers of locally symmetric spaces

Comportement des nombres de Betti des espaces localement symétriques

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ABSTRACT

We announce new results concerning the asymptotic behavior of the Betti numbers of higher rank locally symmetric spaces as their volumes tend to infinity. Our main theorem is a uniform version of the Lück Approximation Theorem (Lück, 1994 [10]) which is much stronger than the linear upper bounds on Betti numbers given by Gromov in Ballmann et al. (1985) [3].

The basic idea is to adapt the theory of local convergence, originally introduced for sequences of graphs of bounded degree by Benjamini and Schramm, to sequences of Riemannian manifolds. Using rigidity theory we are able to show that when the volume tends to infinity, the manifolds locally converge to the universal cover in a sufficiently strong manner that allows us to derive the convergence of the normalized Betti numbers.

RÉSUMÉ

Nous annonçons de nouveaux résultats concernant le comportement asymptotique des nombres de Betti des espaces localement symétriques de rang supérieur lorsque leurs volumes tendent vers l'infini. Notre résultat principal – une version uniforme du théorème d'approximation de Lück (1994) [10] – est plus fort que la majoration linéaire en le volume obtenue par Gromov dans Ballmann et al. (1985) [3].

L'idée de base est d'adapter la théorie de la convergence locale, initialement introduite pour les suites de graphes de degré borné par Benjamimi et Schramm, à des suites de variétés riemanniennes. L'utilisation de théorèmes de rigidité nous permet de montrer que lorsque le volume tend vers l'infini, les variétés convergent localement vers le revêtement universel de manière assez forte pour en déduire la convergence des nombres de Betti normalisés par le volume.

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Soit G un groupe de Lie simple de rang réel supérieur à 2 et soit $X = G/K$ l'espace symétrique associé. Une X-variété est une variété riemannienne complète localement isométrique à X, autrement dit une variété de la forme $M = \Gamma\backslash X$, où $\Gamma \leqslant G$ est un sous-groupe discret sans torsion. On note $b_k(M)$ le k-ième nombre de Betti de M et $\beta_k(X)$ le k-ième nombre de Betti L^2 de X.

Théorème 0.1. *Soit (M_n) une suite de X-variétés compactes dont le rayon d'injectivité est uniformément minoré par une constante strictement positive et telle que* $\mathrm{vol}(M_n) \to \infty$. *Alors on a :*

$$\lim_{n\to\infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} = \beta_k(X)$$

pour $0 \leqslant k \leqslant \dim(X)$.

Margulis conjecture qu'il existe une constante strictement positive qui minore le rayon d'injectivité de toute X-variété compacte (voir [11, p. 322] et [9, Section 10]). Le Théorème 0.1 s'applique donc conjecturalement à toute suite infinie de X-variétés compactes distinctes.

Nous montrons d'ailleurs une version faible de la conjecture de Margulis : pour toute suite (M_n) de X-variétés compactes distinctes et pour tout réel $r > 0$ on a

$$\lim_{n\to\infty} \frac{\mathrm{vol}((M_n)_{<r})}{\mathrm{vol}(M_n)} = 0$$

où $M_{<r}$ désigne l'ensemble des points de M où le rayon d'injectivité local est strictement inférieur à r (voir Corollary 2.5 ci-dessous).

On donne une définition analytique des nombres de Betti L^2 de X dans le paragraphe 2.4 ci-dessous. On peut également les définir comme suit : soit X^* le dual compact de X muni de la métrique Riemannienne induite par la forme de Killing sur $\mathrm{Lie}(G)$. Alors :

$$\beta_k(X) = \begin{cases} 0, & k \neq \frac{1}{2}\dim X, \\ \frac{\chi(X^*)}{\mathrm{vol}(X^*)}, & k = \frac{1}{2}\dim X. \end{cases}$$

Noter que $\chi(X^*) = 0$ sauf si $\mathrm{rank}_{\mathbb{C}}(G) = \mathrm{rank}_{\mathbb{C}}(K)$.

L'idée essentielle pour la démonstration du Théorème 0.1 est d'adapter aux X-variétés la notion de « convergence locale » due à Benjamini et Schramm. On montre alors qu'en rang supérieur toute suite de X-variétés distinctes de volume fini converge localement vers X. D'autre part, en utilisant la définition analytique des nombres de Betti on montre (cette fois sans hypothèse sur le rang de X) que la convergence locale implique la convergence des nombres de Betti normalisés par le volume (sous l'hypothèse d'un rayon d'injectivité uniformément minoré).

Le Théorème 0.1 est faux lorsque le rang réel de G est égal à 1. Cependant, dans [1] nous démontrerons une version affaiblie du Théorème 0.1 valable indépendamment du rang réel de G ainsi qu'une version forte, également valable indépendamment du rang réel de G, mais seulement pour les X-variétés « de congruence » associées à une structure algébrique donnée de G.

1. Introduction

Let G be a simple Lie group with $\mathbb{R}$-rank at least two and let $X = G/K$ be the associated symmetric space. An X-manifold is a complete Riemannian manifold locally isomorphic to X, i.e. a manifold of the form $M = \Gamma\backslash X$ where $\Gamma \leqslant G$ is a discrete torsion free subgroup. We denote by $b_k(M)$ the kth Betti number of M and by $\beta_k(X)$ the kth L^2-Betti number of X.

Theorem 1.1. *Let (M_n) be a sequence of closed X-manifolds with injectivity radius uniformly bounded away from* 0, *and* $\mathrm{vol}(M_n) \to \infty$. *Then*:

$$\lim_{n\to\infty} \frac{b_k(M_n)}{\mathrm{vol}(M_n)} = \beta_k(X)$$

for $0 \leqslant k \leqslant \dim(X)$.

Margulis conjectured that there is a uniform lower bound for the injectivity radius of closed X-manifolds (see [11, p. 322] and [9, Section 10]). If this were true, Theorem 1.1 would apply to any sequence of distinct, closed X-manifolds. (Note that there are only countably many closed X-manifolds and their volumes form a discrete subset of $\mathbb{R}^{\geqslant 0}$.) In this direction, our methods prove the following probabilistic variant of the Margulis conjecture, see Corollary 2.6: For every sequence (M_n) of distinct, closed X-manifolds and $r > 0$ we have $\lim_{n\to\infty} \frac{\mathrm{vol}((M_n)_{<r})}{\mathrm{vol}(M_n)} = 0$ where $M_{<r}$ denotes the set of points in M where the local injectivity radius is less than r.

We will define the L^2-Betti numbers of X analytically in Section 2.4 below, but one can also describe them as follows. Let X^* denote the compact dual of X and recall that the Killing form on $\mathrm{Lie}(G)$ induces a Riemannian structure and a volume form on X^* as well as on X. Then

$$\beta_k(X) = \begin{cases} 0, & k \neq \frac{1}{2}\dim X, \\ \frac{\chi(X^*)}{\mathrm{vol}(X^*)}, & k = \frac{1}{2}\dim X. \end{cases}$$

Note also that $\chi(X^*) = 0$ unless $\mathrm{rank}_{\mathbb{C}}(G) = \mathrm{rank}_{\mathbb{C}}(K)$.

In [1], we will extend our main theorem to semi-simple Lie groups. Also, we shall put the main result in the more general context of counting multiplicities over the unitary dual $\widehat{G}$. Moreover, for *congruence covers* of a fixed arithmetic X-manifold we will obtain even stronger results on the asymptotic behavior of Betti numbers, independently of the $\mathbb{R}$-rank of G. In particular, Theorem 1.1 holds for any sequence of congruence covers.

It is easy to see that Theorem 1.1 is false for rank one symmetric spaces. For instance, suppose that M is a closed hyperbolic n-manifold such that $\pi_1(M)$ surjects on the free group of rank 2. Then finite covers of M corresponding to subgroups of $\mathbb{Z} * \mathbb{Z}$ have first Betti numbers that grow linearly with volume. However, there will be sublinear growth of Betti numbers in any sequence of covers corresponding to a chain of finite index normal subgroups of $\pi_1(M)$ with trivial intersection, by [10]. In [1], we shall give a weaker version of Theorem 1.1 for rank one spaces.

2. The main ideas and skeleton of the proof

2.1. Local convergence of G-spaces

In [5], Benjamini and Schramm introduced a notion of probabilistic, or *local*, convergence for sequences of finite graphs. Here, we introduce a variant of local convergence for spaces modeled on a fixed Lie group.

Let G be a Lie group and let Sub_G denote the space of closed subgroups of G equipped with the Chabauty topology [8]. As a topological space, Sub_G is compact and G acts on Sub_G continuously by conjugation.

Definition 2.1. An *invariant random subgroup* (IRS) of G is a G-invariant probability measure on Sub_G.

This notion has been introduced in [2]. It follows from the compactness of Sub_G that the space of invariant random subgroups of G equipped with the weak* topology is also compact.

For a continuous measure preserving action of G on a probability space the stabilizer of a uniform random point, i.e. the push-forward measure under the map which assigns to each point its stabilizer group, is an IRS. As an example, if $\Gamma < G$ is a lattice then G acts continuously on the space $\Gamma\backslash G$. This action preserves a unique probability measure (Haar measure), and we let μ_Γ be the stabilizer of a random point. Note that the measure μ_Γ is supported on the conjugacy class of Γ. We also let μ_{id} be the measure supported on the trivial subgroup of G and μ_G be the measure supported on the element $G \in \mathrm{Sub}_G$.

Using Borel's density theorem [7] (see also [13]), one can prove that if G is simple then every ergodic IRS other than μ_G is supported on discrete subgroups of G. It is well-known that for discrete subgroups $\Lambda < G$, convergence in the Chabauty topology is equivalent to Gromov Hausdorff convergence of the quotient manifolds $\Lambda\backslash G$. Convergence of invariant random subgroups then has the following interpretation, which is an exact analogue of Benjamini–Schramm convergence of finite graphs:

Lemma 2.2 *(Local convergence). Assume that G is equipped with some left invariant Riemannian metric. Suppose that $\mu_1, \mu_2, \ldots, \mu_\infty$ are IRSs supported on discrete subgroups of a Lie group G. Then the following are equivalent*:

(i) *μ_n converges weakly to μ_∞.*
(ii) *For every pointed Riemannian manifold (M, p) and $R > 0$, and arbitrarily small $\varepsilon, \delta > 0$, the probability that for a μ_n-random $\Gamma \in \mathrm{Sub}_G$ the pointed ball $(B_{\Gamma\backslash G}([\mathrm{id}], R), [\mathrm{id}])$ is $(1+\delta, \varepsilon)$-quasi-isometric to $(B_M(p, R), p)$ converges to the μ_∞-probability of this event.*

2.2. Local convergence in higher rank

Suppose now that G is a simple Lie group with $\mathbb{R}$-rank at least 2 and associated symmetric space $X = G/K$.
In this case, one can classify all the ergodic invariant random subgroups of G.

Proposition 2.3. *Every ergodic invariant random subgroup of G is equal to either μ_{id}, μ_G or to μ_Γ for some lattice Γ in G.*

In general a non-ergodic IRS of G can be obtained as a convex sum of countably many ergodic ones as above.

Proposition 2.3 is implicitly contained in [13, Section 3]. The proof relies on the rigidity theory of higher rank lattices: in particular, on the Stück–Zimmer theorem [13, Theorem 2.1], Borel's Density theorem [7] and the Margulis normal subgroup theorem [11, Chapter VIII].

Using Proposition 2.3 and a quantitative version of the Howe–Moore theorem due to Howe and Oh [12], one can prove the following key result:

Theorem 2.4. *μ_{id} is the only accumulation point of $\{\mu_\Gamma \mid \Gamma$ is a lattice in $G\}$.*

The geometric interpretation of weak convergence given in Theorem 2.2 implies that if $\Gamma_n < G$ is a sequence of lattices such that $\mu_{\Gamma_n} \to \mu_{\mathrm{id}}$, then the injectivity radius of $\Gamma_n\backslash X$ at a random point goes to infinity asymptotically almost surely. Therefore, Theorem 2.4 has the following corollary:

Corollary 2.5. *Suppose that G is a simple Lie group with $\mathbb{R}$-rank at least 2 and associated symmetric space $X = G/K$. If M_n is a sequence of distinct, finite volume X-manifolds, then for every $r > 0$ we have*

$$\lim_{n\to\infty} \frac{\mathrm{vol}((M_n)_{<r})}{\mathrm{vol}(M_n)} = 0,$$

where $M_{<r} := \{x \in M \mid \mathrm{InjRad}_M(x) < r\}$ is the r-thin part of M.

2.3. The Laplacian and heat kernel on differential forms

As before, let $X = G/K$ be the symmetric space associated to a simple Lie group. Denote by $e^{-t\Delta_k^{(2)}}(x, y)$ the heat kernel on L^2 k-forms. The corresponding bounded integral operator in $\mathrm{End}(\Omega^k_{(2)}(X))$ defined by $(e^{-t\Delta_k^{(2)}} f)(x) = \int_X e^{-t\Delta_k^{(2)}}(x, y) f(y)\,\mathrm{d}y,\ \forall f \in \Omega^k_{(2)}(X)$, is the fundamental solution of the heat equation (cf. [4]).

A standard result from local index theory (see e.g. [6, Lemma 3.8]) implies:

Corollary 2.6. *Let $m > 0$. There exists a positive constant $c_1 = c_1(G, m)$ such that*

$$\left\| e^{-t\Delta_k^{(2)}}(x, y)\right\| \leqslant c_1 t^{-d/2} e^{-d(x,y)^2/5t}, \quad 0 < t \leqslant m.$$

Now let $M = \Gamma\backslash X$ be a compact X-manifold. Let Δ_k be the Laplacian on differentiable k-forms on M. It is a symmetric, positive definite, elliptic operator with pure point spectrum. Write $e^{-t\Delta_k}(x, y)$ $(x, y \in M)$ for the heat kernel on k-forms on M, then for each positive t we have:

$$e^{-t\Delta_k}(x, y) = \sum_{\gamma\in\Gamma} (\gamma_{\tilde{y}})^* e^{-t\Delta_k^{(2)}}(\tilde{x}, \gamma\tilde{y}), \tag{1}$$

where $\tilde{x}, \tilde{y}$ are lifts of x, y to X and by $(\gamma_y)^*$, we mean pullback by the map $(x, y) \mapsto (\tilde{x}, \gamma\tilde{y})$. The sum converges absolutely and uniformly for $\tilde{x}, \tilde{y}$ in compacta; this follows from Corollary 2.6, together with the following estimate:

$$\left|\left\{\gamma \in \Gamma\colon d(\tilde{x}, \gamma\tilde{y}) \leqslant r\right\}\right| \leqslant c_2 e^{c_2 r}\, \mathrm{InjRad}_M(x)^{-d}. \tag{2}$$

Here, $c_2 = c_2(G)$ is some positive constant and $d = \dim(X)$.

2.4. (L^2-)Betti numbers

The trace of the heat kernel $e^{-t\Delta_k^{(2)}}(x, x)$ on the diagonal is independent of $x \in X$, being G-invariant. We denote it by

$$\mathrm{Tr}\, e^{-t\Delta_k^{(2)}} := \mathrm{tr}\, e^{-t\Delta_k^{(2)}}(x, x)$$

and set

$$\beta_k(X) := \lim_{t\to\infty} \mathrm{Tr}\, e^{-t\Delta_k^{(2)}}.$$

Recall also that the usual Betti numbers of M are given by

$$b_k(M) = \lim_{t\to\infty} \int_M \mathrm{tr}\, e^{-t\Delta_k}(x, x)\,\mathrm{d}x.$$

The following is a consequence of Corollary 2.6 and Eqs. (1) and (2):

Lemma 2.7. *Let $m > 0$ be a real number. There exists a constant $c = c(m, G)$ such that for any $x \in M$ and $t \in (0, m]$,*

$$\left|\operatorname{tr} e^{-t\Delta_k}(x,x) - \operatorname{Tr} e^{-t\Delta_k^{(2)}}\right| \leqslant c \cdot \operatorname{InjRad}_M(x)^{-d}.$$

2.5. Concluding the proof of Theorem 1.1

Let now $M_n = \Gamma_n \backslash X$ as in the statement of 1.1. Since the injectivity radius is uniformly bounded away from 0 it follows from Corollary 2.5 that $\frac{1}{\operatorname{vol}(M_n)} \int_{(M_n)_{<r}} \operatorname{InjRad}_{M_n}(x)^{-d}\,\mathrm{d}x \to 0$ for every $r > 0$. Hence by Lemma 2.7:

$$\frac{1}{\operatorname{vol}(M_n)} \int\limits_{M_n} \operatorname{tr} e^{-t\Delta_k^{M_n}}(x,x)\,\mathrm{d}x \to \operatorname{tr} e^{-t\Delta_k^{(2)}}$$

uniformly for t on compact subintervals of $(0, \infty)$. Since each term in the limit above is decreasing as a function of t, we deduce that

$$\limsup_{n\to\infty} \frac{b_k(M_n)}{\operatorname{vol}(M_n)} \leqslant \beta_k(X).$$

Finally, using that the usual Euler characteristic is equal to its L^2 analogue and that $\Delta_k^{(2)}$ has zero kernel if $k \neq \frac{1}{2}\dim X$ we derive Theorem 1.1.

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