Functional Analysis/Probability Theory

Geometry of log-concave ensembles of random matrices and approximate reconstruction

Géométrie des ensembles log-concave des matrices aléatoires et une reconstruction approximative

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\textbf{Abstract}

We study the Restricted Isometry Property of a random matrix $\Gamma$ with independent isotropic log-concave rows. To this end, we introduce a parameter $\Gamma_{k,m}$ that controls uniformly the operator norm of sub-matrices with $k$ rows and $m$ columns. This parameter is estimated by means of new tail estimates of order statistics and deviation inequalities for norms of projections of an isotropic log-concave vector.

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\textbf{Résumé}

On étudie la propriété d’isométrie restreinte d’une matrice aléatoire $\Gamma$ dont les lignes sont des vecteurs aléatoires indépendants isotropes log-concave. Pour cela on introduit un paramètre $\Gamma_{k,m}$ qui contrôle uniformément les normes d’opérateurs des sous-matrices de $k$ lignes et $m$ colonnes. Ce paramètre est estimé à l’aide de nouvelles inégalités de queue des statistiques d’ordre et d’inégalités de déviation des normes de projections d’un vecteur aléatoire log-concave.

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\section{Introduction}

Let $T \subset \mathbb{R}^N$ and $\Gamma$ be an $n \times N$ matrix. Consider the problem of reconstructing any vector $x \in T$ from the data $\Gamma x \in \mathbb{R}^n$, with a fast algorithm. Clearly one needs some a priori hypothesis on the subset $T$ and of course, the matrix $\Gamma$ should be suitably chosen. The common and useful hypothesis is that $T$ consists of sparse vectors, that is vectors with short support.
In that setting, Compressed Sensing provides a way of reconstructing the original signal $x$ from its compression $\Gamma x$ with $n \ll N$ by the so-called $\ell_1$-minimization method. The problem of reconstruction can be reformulated after D. Donoho [10] in a language of high-dimensional geometry, namely, in terms of neighborliness of polytopes obtained by taking the convex hull of the columns of $\Gamma$. In this spirit, the sensing matrix is described by its columns. From another point of view, the matrix $\Gamma$ may be also determined by measurements, e.g. by its rows.

Let $0 \leq m \leq N$. Denote by $U_m$ the subset of unit vectors in $\mathbb{R}^N$, which are $m$-sparse, i.e. have at most $m$ non-zero coordinates. The natural scalar product, the Euclidean norm and the unit sphere are denoted by $\langle \cdot, \cdot \rangle$, $| \cdot |$ and $S^{N-1}$. We also denote by the same notation $| \cdot |$ the cardinality of a set. For any $x = (x_i) \in \mathbb{R}^N$ we let $\|x\|_{\infty} = \max_i |x_i|$. By $C$, $C_1$, $c$, etc. we will denote absolute positive constants.

Let $\delta_m = \delta_m(\Gamma) = \sup_{x \in U_m} |\langle \Gamma x, x \rangle|^2 - \mathbb{E}|\langle \Gamma x, x \rangle|^2|$ be the Restricted Isometry Property (RIP) parameter of order $m$. This concept was introduced by E. Candès and T. Tao in [9] and its important feature is that if $\delta_{2m}$ is appropriately small then every $m$-sparse vector $x$ can be reconstructed from its compression $\Gamma x$ by the $\ell_1$-minimization method. The goal now is to check this property for certain models of matrices.

The articles [1–5] considered random matrices with independent columns, and investigated high-dimensional geometric properties of the convex hull of the columns and the RIP for various models of matrices, including the log-concave Ensemble build with independent isotropic log-concave columns. It was shown that various properties of random vectors can be efficiently studied via operator norms and the parameter $\Gamma_{n,m}$ recalled below. In order to control this parameter an efficient technique of chaining was developed in [3] and [4].

In [14], the authors studied the RIP and more generally the parameter $\delta_T = \sup_{x \in T} |\langle \Gamma x, x \rangle|^2 - \mathbb{E}|\langle \Gamma x, x \rangle|^2|$ for random matrices with independent isotropic subgaussian rows. It is natural to ask whether random matrices with independent isotropic log-concave rows also have the RIP.

Fix integers $n, N \geq 1$. Let $Y_1, \ldots, Y_n$ be independent random vectors in $\mathbb{R}^N$ and let $\Gamma$ be the $n \times N$ random matrix with rows $Y_i$. Let $T \subset S^{N-1}$ and $1 \leq k \leq n$ and define the parameter $\Gamma_k(T)$ by

$$\Gamma_k(T)^2 = \sup_{y \in T} \sum_{I \subset \{1, \ldots, n\}, |I| = k} \left| \sum_{i \in I} \langle Y_i, y \rangle \right|^2.$$  \hspace{0.3cm} (1)

We also denote $\Gamma_{k,m} = \Gamma_k(U_m)$. The role of this parameter with respect to the RIP is revealed by the following lemma which reduces a concentration inequality to a deviation inequality:

**Lemma 1.** Let $Y_1, \ldots, Y_n$ be independent isotropic random vectors in $\mathbb{R}^N$. Let $T \subset S^{N-1}$ be a finite set. Let $0 < \theta < 1$ and $B > 1$. Then with probability at least $1 - |T| \exp(-3\theta^2 n/8B^2)$ one has

$$\sup_{y \in T} \left| \frac{1}{n} \sum_{i=1}^n \left( |\langle Y_i, y \rangle|^2 - \mathbb{E}|\langle Y_i, y \rangle|^2 \right) \right| \leq \theta + \frac{1}{n} (\Gamma_k(T)^2 + \mathbb{E}\Gamma_k(T)^2),$$

where $k \leq n$ is the largest integer satisfying $k \leq \left( \Gamma_k(T)/B \right)^2$.

In this note we focus on the compressed sensing setting where $T$ is the set of sparse vectors. Lemma 1 shows that after a suitable discretization, checking the RIP reduces to estimating $\Gamma_{k,m}$. The idea of such an approach, when $k = n$, originated from the work of J. Bourgain [8] on the empirical covariance matrix. It was developed in [3] and [5] (with $T = U_m$), where the estimate of $\Gamma_{n,m}$ played a central role for solving the Kannan–Lovász–Simonovits conjecture related to complexity of computing high-dimensional volumes [11]; and it was studied in [13], where $\Gamma_k(T)$ was estimated by means of Talagrand $\gamma$-functionals.

Using Lemma 1 it can be shown (cf. [5] for a similar argument) that if $0 < \theta < 1$, $B \geq 1$, and $m \leq N$ satisfies $m \log(CN/m) \leq 3\theta^2 n/16B^2$, then with probability at least $1 - \exp(-\theta^2 n/16B^2)$ one has

$$\delta_m(\Gamma/\sqrt{n}) = \sup_{y \in U_m} \left| \frac{1}{n} \sum_{i=1}^n \left( |\langle Y_i, y \rangle|^2 - \mathbb{E}|\langle Y_i, y \rangle|^2 \right) \right| \leq C \theta + \frac{C}{n} (\Gamma_{k,m}^2 + \mathbb{E}\Gamma_{k,m}^2),$$  \hspace{0.3cm} (2)

where $k \leq n$ is the largest integer satisfying $k \leq \left( \Gamma_{k,m}/B \right)^2$ (note that $k$ is a random variable).

We consider the log-concave Ensemble of $n \times N$ matrices with independent isotropic log-concave rows. Recall that a random vector is isotropic log-concave if it is centered, its covariance matrix is the identity and its distribution has a log-concave density. Our goal is to bound $\Gamma_{k,m}$ for this Ensemble. This leads to questions that require a deeper understanding of some geometric parameters of log-concave measures, such as tail estimates for order statistics and deviation inequalities for norms of projections. Proofs and related results will be presented in [6].

**2. Main results**

Our main result, Theorem 6, provides upper estimates for $\Gamma_{k,m}$ valid with large probability for matrices from the log-concave Ensemble. To achieve this we need some intermediate steps also of a major importance. For a random vector $X$ and $p > 0$, we define the following natural parameter:
\[ \sigma_X(p) = \sup_{t \in \mathbb{S}^{N-1}} (\mathbb{E}|t, X|^p)^{1/p}. \]

It is known that \( \sigma_X(p) \leq p \) for isotropic log-concave \( X \) and \( p \geq 2 \). Paouris’ Theorem ([15]) states
\[ \left( \mathbb{E}|X|^p \right)^{1/p} \leq C \left( \left( \mathbb{E}|X|^2 \right)^{1/2} + \sigma_X(p) \right). \]

It is a consequence of Theorem 8.2 combined with Lemma 3.9 in [15], note that Lemma 3.9 holds not only for convex bodies but for log-concave measures as well.

We extend the Paouris Theorem to the following bound on deviations of norm of projections of an isotropic log-concave vector, uniform over all coordinate projections \( P_1 \) of a fixed rank:

**Theorem 2.** Let \( m \leq N \) and \( X \) be an isotropic log-concave vector in \( \mathbb{R}^N \). Then for every \( t \geq 1 \) one has
\[ \mathbb{P} \left( \sup_{|I| = m} |P_1 X| \geq Ct \sqrt{m \log \left( \frac{eN}{m} \right)} \right) \leq \exp \left( -t \frac{\sqrt{m}}{\sqrt{\log(em)}} \log \left( \frac{eN}{m} \right) \right). \]

This theorem is sharp up to \( \sqrt{\log(em)} \) in the probability estimate as the case of a vector with independent exponential coordinates shows. Actually our further applications require a stronger result in which the bound for probability is improved by involving the parameter \( \sigma_X \) and its inverse \( \sigma_X^{-1} \), namely

**Theorem 3.** Let \( m \leq N \) and \( X \) be an isotropic log-concave vector in \( \mathbb{R}^N \). Then for any \( t \geq 1 \),
\[ \mathbb{P} \left( \sup_{|I| = m} |P_1 X| \geq Ct \sqrt{m \log \left( \frac{eN}{m} \right)} \right) \leq \exp \left( -\sigma_X^{-1} \left( t \sqrt{m \log(eN/m)} \right) \right), \]
where \( m_0 = m_0(X, t) = \sup (k \leq m: k \log(eN/k) \leq \sigma_X^{-1} (t \sqrt{m \log(eN/m)})). \)

Theorem 3 is based on tail estimates for order statistics of isotropic log-concave vectors. By \( (X^*(i))_1 \) we denote the non-increasing rearrangement of \( (|X(i)|)_1 \). Combining (3) with methods of [12] we obtain

**Theorem 4.** Let \( X \) be an \( N \)-dimensional isotropic log-concave vector. Then for every \( t \geq C \log(eN/\ell) \),
\[ \mathbb{P}(X^*(\ell) \geq t) \leq \exp(-\sigma_X^{-1}(C^{-1} t \sqrt{\ell})). \]

Introduction of the parameter \( \sigma_X \) enables us to obtain new inequalities for convolutions of log-concave measures. Let \( X_1, \ldots, X_n \) be independent isotropic log-concave random vectors in \( \mathbb{R}^N \). We will consider weighted sums of the vectors \( X_i \) of the form \( Y = \sum_{i=1}^n \psi_i X_i \), where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \). Bernstein’s inequality and \( \psi_i \) estimate for isotropic log-concave random vectors give \( \sigma_Y(p) \leq C(\sqrt{p} |x| + p \|x\|_\infty) \) for \( p \geq 1 \). Together with Theorem 3 this yields the following:

**Corollary 5.** Assume that \( |x| \leq 1 \) and \( b \geq 1 \), then \( \max(\|x\|_\infty, 1/\sqrt{m}) \). Then for any \( t \geq 1 \),
\[ \mathbb{P} \left( \sup_{|I| = m} |P_1 Y| \geq Ct \sqrt{m \log \left( \frac{eN}{m} \right)} \right) \leq \exp \left( -\frac{t \sqrt{m \log(eN/m)}}{b \log(e^2b^2m)} \right). \]

We now pass to bounds on deviation of \( I_{k,m}^\star \). To get a slightly simplified formula we assume that \( N \geq n \).

**Theorem 6.** Let \( 1 \leq n \leq N \), and let \( \Gamma^\star \) be an \( n \times N \) random matrix with independent isotropic log-concave rows. For any integers \( k \leq n \), \( m \leq N \) and any \( t \geq 1 \), we have
\[ \mathbb{P}(I_{k,m}^\star \geq Ct \lambda) \leq \exp \left( -t \lambda / \sqrt{\log(3m)} \right), \]
where \( \lambda = \sqrt{\log(3m)} \sqrt{m \log(eN/m)} + \sqrt{k} \log(en/k) \).

The threshold value \( \lambda \) in Theorem 6 is optimal, up to the factor of \( \sqrt{\log(3m)} \). Assuming additionally unconditionality of the distributions of the rows, we can remove this factor and get a sharp estimate ([17]).

The proof of the above theorem is composed of two parts, depending on the relation between \( k \) and the quantity \( k' = \inf(\ell \geq 1: m \log(eN/m) \leq \ell \log(en/\ell)) \). First, we adjust the chaining argument from [3] to reduce the problem to the
case \( k \leq k' \). This step also involves Theorem 2. Next, we use Corollary 5 combined with another chaining to complete the argument.

Theorem 6 together with (2) allows us to prove the RIP result for matrices \( \Gamma' \) with independent isotropic log-concave rows. The result is optimal, up to the factor \( \log \log 3m \), as shown in [4]. As for Theorem 6, assuming unconditionality of the distributions of the rows, we can remove this factor ([7]).

**Theorem 7.** Let \( 0 < \theta < 1, \ 1 \leq n \leq N \). Let \( \Gamma \) be an \( n \times N \) random matrix with independent isotropic log-concave rows. There exists \( c(\theta) > 0 \) such that \( \delta_m(\Gamma/\sqrt{n}) \leq \theta \) with overwhelming probability whenever

\[
m \log^2(2N/m) \log \log 3m \leq c(\theta)n.
\]

**References**


