



Probability Theory/Mathematical Physics

Ghirlanda–Guerra identities and ultrametricity: An elementary proof in the discrete case

Ultramétrie et identités de Ghirlanda–Guerra : une preuve élémentaire du cas discret

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ABSTRACT

In this Note we give another proof of the fact that a random overlap array, which satisfies the Ghirlanda–Guerra identities and whose elements take values in a finite set, is ultrametric with probability one. The new proof bypasses random change of density invariance principles for directing measures of such arrays and, in addition to the Dobysh–Sudakov representation, is based only on elementary algebraic consequences of the Ghirlanda–Guerra identities.

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R É S U M É

Dans cette Note nous donnons une nouvelle preuve du fait qu'une matrice aléatoire infinie, qui satisfait l'identité Ghirlanda–Guerra et dont les coefficients prennent leurs valeurs dans un ensemble fini, est ultramétrique avec probabilité un. La preuve utilise uniquement des conséquences algébriques élémentaires des identités Ghirlanda–Guerra et la représentation de Dobysh–Sudakov.

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1. Introduction and main result

In this Note we will give a simplified proof of the main result in [5]. Let us consider an infinite random array $R = (R_{l,l'})_{l,l' \geq 1}$ which is symmetric, non-negative definite and weakly exchangeable, i.e. for any $n \geq 1$ and for any permutation ρ of $\{1, \dots, n\}$ the matrix $(R_{\rho(l), \rho(l')})_{l,l' \leq n}$ has the same distribution as $(R_{l,l'})_{l,l' \leq n}$. We assume that the diagonal elements are identically 1, $R_{l,l} = 1$ and that the non-diagonal elements take finitely many values,

$$\mathbb{P}(R_{1,2} = q_l) = p_l \quad (1)$$

for some $-1 \leq q_1 < q_2 < \dots < q_k \leq 1$ and $p_l > 0$, $p_1 + \dots + p_k = 1$. The array R is said to satisfy the Ghirlanda–Guerra identities [4] if for any $n \geq 2$ and any bounded measurable functions $f = f((R_{l,l'})_{l,l' \leq n})$ and $\psi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\mathbb{E} f \psi(R_{1,n+1}) = \frac{1}{n} \mathbb{E} f \mathbb{E} \psi(R_{1,2}) + \frac{1}{n} \sum_{l=2}^n \mathbb{E} f \psi(R_{1,l}). \quad (2)$$

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By the positivity principle of Talagrand [8,10], the Ghirlanda–Guerra identities imply that $R_{1,2} \geq 0$ with probability one and, therefore, we can assume that $q_1 \geq 0$. The following result was proved in [5]:

Theorem 1.1. *Under assumptions (1) and (2), the array R is ultrametric,*

$$\mathbb{P}(R_{2,3} \geq \min(R_{1,2}, R_{1,3})) = 1. \tag{3}$$

Another way to express the event in (3) is to say that

$$R_{1,2} \geq q_l, R_{1,3} \geq q_l \implies R_{2,3} \geq q_l \text{ for all } 1 \leq l \leq k. \tag{4}$$

Infinite arrays that satisfy the Ghirlanda–Guerra identities arise as the limits of the overlap arrays in the Sherrington–Kirkpatrick spin glass models (see e.g. [10,7]). The assumption (1) is purely technical (and unfortunately is not satisfied in the most important situations). The first ultrametricity result was proved in [2] under different conditions which also included (1), but instead of (2) the authors worked with the Aizenman–Contucci stochastic stability [1]. The original proof of Theorem 1.1 in [5] utilized a key idea from [2], namely, the existence of directing measures guaranteed by the Dobysh–Sudakov representation result in [3], and we will still rely on this representation here. However, we will completely avoid proving any invariance principles under random changes of density for the directing measure, which played crucial roles both in [2] and [5] and our new induction will be quite elementary in nature. M. Talagrand gave a proof of Theorem 1.1 in [9] that did not use the Dobysh–Sudakov representation but still used the invariance principle from [5]. The Dobysh–Sudakov representation [3] (for detailed proof see [6]) states that given a symmetric, non-negative definite and weakly exchangeable array R , there exists a random measure μ on $H \times [0, \infty)$, where H is a separable Hilbert space, such that R is equal in distribution to the array

$$(\sigma^l \cdot \sigma^{l'} + a^l \delta_{l,l'})_{l,l' \geq 1} \tag{5}$$

where (σ^l, a^l) is an i.i.d. sequence from μ and $\sigma \cdot \sigma'$ denotes the scalar product on H . Let us denote by G the marginal of μ on H . The following simple consequence of the Ghirlanda–Guerra identities (2) was proved in Theorem 2 in [5]:

Proposition 1.1. *Under (1) and (2), the random measure G is (countably) discrete and is concentrated on the sphere of radius $\sqrt{q_k}$ with probability one.*

In particular, this implies that $a^l = 1 - q_k$ in (5) and without loss of generality we can redefine the array by $R_{l,l'} = \sigma^l \cdot \sigma^{l'}$ for an i.i.d. sequence (σ^l) from G . Since $R_{l,l'} = q_k$ if and only if $\sigma^l = \sigma^{l'}$, we have

$$\mathbb{P}(R_{1,2} = q_k, R_{1,3} = q_k, R_{2,3} < q_k) = 0,$$

which proves “ultrametricity at the level k ” in the sense of (4). As in [2] and [5], we would like to find a way to make an induction step and prove “ultrametricity at the level $k - 1$ ”. The main new idea of the paper will be to consider the distribution of the array $(R_{l,l'})$ conditionally on the event that all replicas (σ^l) are different and prove that this new distribution is well-defined and satisfies all the conditions of the Dobysh–Sudakov representation. Since on the above event the elements of the new array cannot take value q_k , the induction step will follow.

2. Proof

By Proposition 1.1, $G = \sum_{l \geq 1} w_l \delta_{\xi_l}$ for some random weights (w_l) and random sequence (ξ_l) in H such that $\xi_l \cdot \xi_l = q_k$. Let us denote by $\langle \cdot \rangle$ the average with respect to $G^{\otimes \infty}$ and by \mathbb{E} the expectation with respect to the randomness of G . With these notations, the Ghirlanda–Guerra identities (2) can be rewritten as

$$\mathbb{E}\langle f_n \psi(R_{1,n+1}) \rangle = \frac{1}{n} \mathbb{E}\langle f_n \rangle \mathbb{E}\langle \psi(R_{1,2}) \rangle + \frac{1}{n} \sum_{l=2}^n \mathbb{E}\langle f_n \psi(R_{1,l}) \rangle. \tag{6}$$

For each $n \geq 2$, let us consider the event

$$A_n = \{R_{l,l'} \neq q_k, \forall 1 \leq l < l' \leq n\} \tag{7}$$

and let \mathbb{P}_n be the distribution of the $n \times n$ matrix $R^n = (\sigma^l \cdot \sigma^{l'})_{l,l' \leq n}$ conditionally on A_n ,

$$\mathbb{P}_n(B) = \frac{\mathbb{E}\langle I(R^n \in B) I_{A_n} \rangle}{\mathbb{E}\langle I_{A_n} \rangle}. \tag{8}$$

It is obvious that \mathbb{P}_n is concentrated on the symmetric non-negative definite matrices with off-diagonal elements now taking values $\{q_1, \dots, q_{k-1}\}$ and \mathbb{P}_n is invariant under the permutation of replica indices since the set A_n is. We will now show that \mathbb{P}_{n+1} restricted to the first n replica coordinates coincides with \mathbb{P}_n and, thus, the sequence (\mathbb{P}_n) defines a law of the infinite overlap array.

Lemma 2.1. For any measurable function f of the overlaps on n replicas,

$$\mathbb{E}\langle f(R^n)I_{A_{n+1}} \rangle = (1 - p_k)\mathbb{E}\langle f(R^n)I_{A_n} \rangle. \tag{9}$$

Proof. Notice that $A_n = \{\sigma^1, \dots, \sigma^n$ are all different} by Proposition 1.1 and, therefore,

$$I_{A_{n+1}} = I_{A_n} - \sum_{l \leq n} I_{A_n \cap \{R_{l,n+1} = q_k\}}. \tag{10}$$

This implies that

$$\mathbb{E}\langle f(R^n)I_{A_{n+1}} \rangle = \mathbb{E}\langle f(R^n)I_{A_n} \rangle - \sum_{l \leq n} \mathbb{E}\langle f(R^n)I_{A_n}I(R_{l,n+1} = q_k) \rangle.$$

Using the Ghirlanda–Guerra identities (6), for each $l \leq n$,

$$\begin{aligned} \mathbb{E}\langle f(R^n)I_{A_n}I(R_{l,n+1} = q_k) \rangle &= \frac{p_k}{n} \mathbb{E}\langle f(R^n)I_{A_n} \rangle + \frac{1}{n} \sum_{l' \neq l}^n \mathbb{E}\langle f(R^n)I_{A_n}I(R_{l,l'} = q_k) \rangle \\ &= \frac{p_k}{n} \mathbb{E}\langle f(R^n)I_{A_n} \rangle \end{aligned}$$

since $A_n \subseteq \{R_{l,l'} \neq q_k\}$ and, thus, $I_{A_n}I(R_{l,l'} = q_k) = 0$. Adding up over $l \leq n$ finishes the proof. \square

First, using (9) inductively for $f \equiv 1$ we get $\mathbb{E}\langle I_{A_n} \rangle = (1 - p_k)^{n-1}$ and then dividing (9) by $(1 - p_k)^n$,

$$\frac{\mathbb{E}\langle f(R^n)I_{A_{n+1}} \rangle}{\mathbb{E}\langle I_{A_{n+1}} \rangle} = \frac{\mathbb{E}\langle f(R^n)I_{A_n} \rangle}{\mathbb{E}\langle I_{A_n} \rangle}. \tag{11}$$

This means that the family (\mathbb{P}_n) is consistent and by Kolmogorov’s theorem we can define the distribution of the infinite array with the corresponding marginals given by \mathbb{P}_n . Let us consider an array $Q = (Q_{l,l'})_{l,l' \geq 1}$ with this distribution.

Proof of Theorem 1.1. By construction, Q is a symmetric, non-negative definite and weakly exchangeable array with diagonal elements equal to q_k and off-diagonal elements taking values $\{q_1, \dots, q_{k-1}\}$ with probabilities $\mathbb{P}(Q_{1,2} = q_l) = p_l/(1 - p_k)$. Using the Dobrushin–Sudakov representation for the array Q implies that there exists a random measure G' on H such that Q can be generated as

$$Q_{l,l'} = \sigma^l \cdot \sigma^{l'} + \delta_{l,l'}(q_k - \sigma^l \cdot \sigma^l)$$

for an i.i.d. sequence (σ^l) from G' . Since $\sigma^l \cdot \sigma^{l'} \in \{q_1, \dots, q_{k-1}\}$, it is easy to see that the support of G' must be inside the sphere of radius $\sqrt{q_{k-1}}$ for, otherwise, with positive probability we could sample two points σ^1, σ^2 arbitrarily close to a point σ such that $\|\sigma\| > \sqrt{q_{k-1}}$ which would contradict that $\sigma^1 \cdot \sigma^2 \leq q_{k-1}$ (see [2] or [5] for details). Because of this, the truncated array $(Q_{l,l'} \wedge q_{k-1})_{l,l' \geq 1}$ can be computed as

$$Q_{l,l'} \wedge q_{k-1} = \sigma^l \cdot \sigma^{l'} + \delta_{l,l'}(q_{k-1} - \sigma^l \cdot \sigma^l) \tag{12}$$

and it is non-negative definite as the sum of two non-negative definite arrays. If we recall the definition (8), the matrix $(Q_{l,l'})_{l,l' \leq n}$ is obtained by sampling n configurations from the measure $G = \sum_{l \geq 1} w_l \delta_{\xi_l}$ conditionally on the event that these configurations are different. Since with positive probability we can sample ξ_1, \dots, ξ_n , we must have that the matrix $(\xi_l \cdot \xi_{l'} \wedge q_{k-1})_{l,l' \leq n}$ is non-negative definite and, therefore, $(\xi_l \cdot \xi_{l'} \wedge q_{k-1})_{l,l' \geq 1}$ is non-negative definite with probability one. This of course means that $(R_{l,l'} \wedge q_{k-1})_{l,l' \geq 1}$ is also non-negative definite. Since the function $x \wedge q_{k-1}$ can be approximated by polynomials, the truncated overlap array also satisfies the Ghirlanda–Guerra identities and its elements now take values in $\{q_1, \dots, q_{k-1}\}$. One can proceed by induction on k . \square

Even though we did not need it in the proof, one can show that the measure G' is actually concentrated on the sphere of radius $\sqrt{q_{k-1}}$ by using Proposition 1.1 and the following observation:

Lemma 2.2. The distribution of Q satisfies the Ghirlanda–Guerra identities,

$$\mathbb{E}f(Q^n)\psi(Q_{1,n+1}) = \frac{1}{n}\mathbb{E}f(Q^n)\mathbb{E}\psi(Q_{1,2}) + \frac{1}{n}\sum_{l=2}^n \mathbb{E}f(Q^n)\psi(Q_{1,l}). \tag{13}$$

Proof. The proof is a straightforward computation using (10) and the Ghirlanda–Guerra identities. \square

The idea of the proof of Theorem 1.1 suggests the following criterion of ultrametricity in the general case without the assumption (1). Given $q \in [0, 1]$ such that $\mathbb{P}(R_{1,2} < q) > 0$, consider the events

$$A_{n,q} = \{R_{l,l'} < q, \forall 1 \leq l < l' \leq n\} \quad (14)$$

and let $\mathbb{P}_{n,q}$ be the distribution of $R^n = (R_{l,l'})_{l,l' \leq n}$ conditionally on $A_{n,q}$.

Theorem 2.3. *Under (2), the array R is ultrametric if and only if for any q such that $\mathbb{P}(R_{1,2} < q) > 0$ and any set B of 3×3 matrices such that $\mathbb{P}_{3,q}(R^3 \in B) > 0$ we have $\limsup_{n \rightarrow \infty} \mathbb{P}_{n,q}(R^3 \in B) > 0$.*

One can check that, in one direction, ultrametricity yields a relationship of the type (10) which implies the consistency of the sequence $(\mathbb{P}_{n,q})$ as in Lemma 2.1 and $\mathbb{P}_{n,q}(R^3 \in B) = \mathbb{P}_{3,q}(R^3 \in B)$. In the other direction, for any B with $\mathbb{P}_{3,q}(R^3 \in B) > 0$ we can choose the limit \mathbb{P}_q over a subsequence of $\mathbb{P}_{n,q}$ such that $\mathbb{P}_q(R^3 \in B) > 0$. If ultrametricity fails, one can make a choice of a subset of non-ultrametric configurations B and $q \in [0, 1]$ that will lead to contradiction with the Dvbysh–Sudakov representation for \mathbb{P}_q .

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