Let $n$ be a non-zero natural number, $p$ a prime number, $q$ a power of $p$ and let $\mathbb{F}_q$ denote a finite field with $q$ elements. Let $G_n = \text{SL}_n(\mathbb{F}_q)$, $\text{SL}_n(\mathbb{F}_q)$ or $\text{PGL}_n(\mathbb{F}_q)$, where $q$ is a power of some prime number $p$, let $U$ denote a Sylow $p$-subgroup of $G$ and let $R$ be a commutative ring in which $p$ is invertible. Let $D(U)$ denote the derived subgroup of $U$ and let $e = \frac{1}{|D(U)|} \sum_{u \in D(U)} u$. The aim of this Note is to prove that the $R$-algebras $RG$ and $eRGe$ are Morita equivalent (through the natural functor $RG\text{-mod} \rightarrow eRGe\text{-mod}$, $M \mapsto eM$).

© 2011 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Let $n$ be a non-zero natural number, $p$ a prime number, $q$ a power of $p$ and let $\mathbb{F}_q$ denote a finite field with $q$ elements. Let $G_n = \text{SL}_n(\mathbb{F}_q)$, $\text{SL}_n(\mathbb{F}_q)$ or $\text{PGL}_n(\mathbb{F}_q)$, where $q$ is a power of some prime number $p$, let $U$ denote a Sylow $p$-subgroup of $G$ and let $R$ be a commutative ring in which $p$ is invertible. Let $D(U)$ denote the derived subgroup of $U$ and let $e = \frac{1}{|D(U)|} \sum_{u \in D(U)} u$. The aim of this Note is to prove that the $R$-algebras $RG$ and $eRGe$ are Morita equivalent (through the natural functor $RG\text{-mod} \rightarrow eRGe\text{-mod}$, $M \mapsto eM$).

© 2011 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Let $G = \text{GL}_n(\mathbb{F}_q)$, $\text{SL}_n(\mathbb{F}_q)$ or $\text{PGL}_n(\mathbb{F}_q)$, where $q$ is a power of some prime number $p$, let $U$ denote a Sylow $p$-subgroup of $G$ and let $R$ be a commutative ring in which $p$ is invertible. Let $D(U)$ denote the derived subgroup of $U$ and let $e = \frac{1}{|D(U)|} \sum_{u \in D(U)} u$. The aim of this Note is to prove that the $R$-algebras $RG$ and $eRGe$ are Morita equivalent (through the natural functor $RG\text{-mod} \rightarrow eRGe\text{-mod}$, $M \mapsto eM$).

© 2011 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
Then $e_n$ is an idempotent of $RG_n$. The aim of this Note is to prove the following result (recall that an idempotent $i$ of a ring $A$ is called full if $A = AiA$):

**Theorem 1.** If $p$ is invertible in $R$, then $e_n$ is a full idempotent of $RG_n$.

**Proof.** First, let $R_0 = \mathbb{Z}[1/p]$, let $\zeta$ be a primitive $p$-th root of unity in $\mathbb{C}$ and let $\hat{R}_0 = R_0[\zeta]$. Let $\mathcal{J}_0 = R_0e_nR_0G_n$, $\mathcal{J}_1 = R_0G_n e_n R_0G_n$ and $\mathcal{J} = RG_nRG_n$. Since $p$ is invertible in $R$, there is a unique morphism of rings $R_0 \rightarrow R$ which extends to a morphism of rings $R_0G_n \rightarrow RG_n$. So if $1 \in \mathcal{J}_0$, then $1 \in \mathcal{J}_1$. Also, as $(1, \zeta, \ldots, \zeta^{p−2})$ is an $R_0$-basis of $\hat{R}_0$, it is also an $R_0G_n$-basis of $R\hat{R}_0G_n$. Therefore, if $1 \in \hat{R}_0G_n e_n R_0G_n = \hat{R}_0 \otimes_{R_0} (R_0G_n e_n R_0G_n)$, then $1 \in \mathcal{J}_0$. Consequently, in order to prove Theorem 1, we may (and we shall) work under the following hypothesis:

**Hypothesis.** From now on, and until the end of this proof, we assume that $R = \mathbb{Z}[1/p, \zeta]$.

Now, let $P_n$ denote the subgroup of $\text{SL}_n(\mathbb{F}_q)$ defined by

$$P_n = \left\{ \begin{pmatrix} a_1 & \cdots & a_n \\ M & \end{pmatrix} : a_1, \ldots, a_n \in \mathbb{F}_q, a_n \in \mathbb{F}_q^* \right\}.$$

Then $U_n \subset P_n$. We shall prove by induction on $n$ that $e_n$ is a full idempotent of $RP_n$. ($P_n$)

It is clear that Theorem 1 follows immediately from ($P_n$).

As $e_0 = 1$ and $e_1 = 1$, it follows that ($P_1$) and ($P_2$) hold. So assume that $n \geq 3$ and that ($P_{n−1}$) holds. Let $I_n$ denote the identity $n \times n$ matrix and let

$$V_n = \left\{ \begin{pmatrix} a_1 & \cdots & a_n \\ M & \end{pmatrix} : a_1, \ldots, a_n \in \mathbb{F}_q \right\}.$$

Then $V_n \cong (\mathbb{F}_q^*)^{n−1}$ and $P_n = \text{SL}_{n−1}(\mathbb{F}_q) \times V_n \cong \text{SL}_{n−1}(\mathbb{F}_q) \times (\mathbb{F}_q^*)^{n−1}$. We set $V'_n = D(U_n) \cap V_n$, so that $V'_n \cong (\mathbb{F}_q^*)^{n−2}$ is normalized by $P_{n−1}$. Then

$$D(U_n) = D(U_{n−1}) \times V'_n.$$

We now define $f_n = \frac{1}{|V'_n|} \sum_{v \in V'_n} v$, so that $e_n = e_{n−1} f_n$.

By the induction hypothesis, there exist $g_1, h_1, \ldots, g_i, h_i$ in $P_{n−1}$ and $r_1, \ldots, r_i$ in $R$ such that

$$1 = \sum_{i=1}^l r_i g_i e_{n−1} h_i.$$

Therefore, as $P_{n−1}$ normalizes $V'_n$, it centralizes $f_n$ and so

$$f_n = \left( \sum_{i=1}^l r_i g_i e_{n−1} h_i \right) f_n = \sum_{i=1}^l r_i g_i e_{n−1} f_n h_i = \sum_{i=1}^l r_i g_i e_n h_i.$$

So $f_n \in RP_n e_n RP_n$.

Let $\mu_\rho$ denote the subgroup of $R^\times$ generated by $\zeta$. If $\chi \in \text{Hom}(V_n, \mu_\rho)$, we define $b_\chi$ to be the associated primitive idempotent of $RV_n$:

$$b_\chi = \frac{1}{|V_n|} \sum_{v \in V_n} \chi(v)^{-1} v \in RV_n.$$
Then, as \( V_n \) is an elementary abelian \( p \)-group, we get

\[
f_n = \sum_{\chi \in \text{Hom}(V_n, \mu_p)} b\chi.
\]

We fix a non-trivial element \( \chi_0 \in \text{Hom}(V_n, \mu_p) \) whose restriction to \( V'_n \) is trivial. Then

\[
b\chi_0 = b\chi_0 f_n \quad \text{and} \quad b_1 = b_1 f_n,
\]

so \( b_1 \) and \( b\chi_0 \) belong to \( R P_n e_n R P_n \).

But \( SL_{n-1}(F) \subset P_n \) has only two orbits for its action on \( \text{Hom}(V_n, \mu_p) \) (because \( n - 1 \geq 2 \)): the orbit of 1 and the orbit of \( \chi_0 \). Therefore, \( b\chi \in R P_n e_n R P_n \) for all \( \chi \in \text{Hom}(V_n, \mu_p) \). Consequently,

\[
1 = \sum_{\chi \in \text{Hom}(V_n, \mu_p)} b\chi \in R P_n e_n R P_n,
\]

as desired. \( \Box \)

Finite reductive groups. Let \( F \) be an algebraic closure of \( \mathbb{F}_q \), let \( G \) be a connected reductive group over \( F \) and let \( F : G \rightarrow G \) be an isogeny such that some power \( F^\delta \) is a Frobenius endomorphism relative to an \( \mathbb{F}_q \)-structure. We denote by \( U \) an \( F \)-stable maximal unipotent subgroup of \( G \) (it is the unipotent radical of an \( F \)-stable Borel subgroup). Define

\[
e = \frac{1}{|D(U)^F|} \sum_{u \in D(U)^F} u \in RG^F.
\]

**Theorem 2.** Assume that \((G, F)\) is split of type A. Then \( e \) is a full idempotent of \( RG^F \).

**Proof.** If \((G, F)\) is split of type A, then we may assume that \( \delta = 1 \). Then there is a morphism of groups \( \pi : G_n \rightarrow G^F \) such that the image of \( U_n \) is \( U^F \). As \( D(U^F) = D(U)^F \) in this case, the extension of this morphism to the group algebras \( \tilde{\pi} : RG_n \rightarrow RG^F \) sends \( e_n \) to \( e \). By Theorem 1, the two-sided ideal of \( RG_n \) generated by \( e_n \) contains 1, so the result follows by applying \( \tilde{\pi} \). \( \Box \)

**Corollary 3.** If \((G, F)\) is split of type A, then the functors

\[
RG^F \text{-mod} \quad \longrightarrow \quad eRG^F e \text{-mod} \quad \text{and} \quad RG^F e \text{-mod} \quad \longrightarrow \quad RG^F e \text{-mod}
\]

\[M \quad \longmapsto \quad eM \quad \text{and} \quad N \quad \longmapsto \quad e^* \otimes_{eRG^F e} N\]

are mutually inverse equivalences of categories. In particular, \( RG^F \) and \( eRG^F e \) are Morita equivalent, and \( RG^F e \) is a progenerator for \( RG^F \).

**Proof.** This follows from Theorem 2 and, for instance, [3, Example 18.30]. \( \Box \)

**Example.** Theorem 2 and Corollary 3 can be applied whenever \( G^F = GL_n(\mathbb{F}_q), SL_n(\mathbb{F}_q) \) or \( PGL_n(\mathbb{F}_q) \).

**Comments.** (1) It is natural to ask whether Theorem 2 (or Corollary 3) can be generalized to other finite reductive groups. In fact, it cannot: indeed, if for instance \( R = \mathbb{C} \), then saying that \( e \) is a full idempotent of \( RG^F \) means that every irreducible character of \( G^F \) is an irreducible component of an Harish-Chandra induced of some Gelfand–Graev character. But, if \( G \) is quasi-simple and \((G, F)\) is not split of type A, then \( G^F \) admits a unipotent character which does not belong to the principal series: this character cannot be an irreducible component of an Harish-Chandra induced of a Gelfand–Graev character.

(2) In [1], a crucial step for the proof of a special case of the geometric version of Broué’s abelian defect conjecture was [1, Theorem 4.1], where R. Rouquier and the author have proved the above Theorem 2 in the case where \( R \) is the integral closure of \( \mathbb{Z}_\ell \) in a sufficiently large algebraic extension of \( \mathbb{Q}_\ell \) (here, \( \ell \) is a prime number different from \( p \)). The proof was essentially based on the classification, due to Dipper [2. 4.15 and 5.23], of simple modules for \( G_n \) in characteristic \( \ell \), and especially of cuspidal ones, which involves the Deligne-Lusztig theory.

The interest of the proof given here is that it does not rely on any classification of simple modules, and is based on elementary methods: as a by-product of this elementariness, Theorem 2 and Corollary 3 are valid over any commutative ring (in which \( p \) is invertible, which is a necessary condition if one wants the idempotent \( e_n \) to be well-defined).

**References**