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Combinatorics

On the edge Szeged index of bridge graphs

Sur l'indice de Szedge d'arête de graphes pontés

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ABSTRACT

In this Note, we introduce a formula for the edge Szeged index of bridge graphs. Using this formula, the edge Szeged indices of several graphs are computed.

RÉSUMÉ

Nous présentons une formule pour l'indice de Szedge pour les arêtes de graphes pontés. Nous calculons également, grâce à cette formule, les indices de Szedge d'arête de plusieurs graphes.

1. Introduction and preliminaries

Topological indices are numerical graph invariants that quantitatively characterize molecular structure. The Wiener index is one of the oldest topological indices [13,15]. The Szeged index is closely related to the Wiener index.

We consider simple graphs. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, denote by $d_G(u,v)$ or $d(u,v)$ the distance between vertices u and v in G, i.e., the length of a shortest path in G connecting u and v. If e is an edge of G connecting vertices u and v, then we write $e = uv$ or $e = vu$. For $e = uv \in E(G)$, let $n_u(e|G)$ and $n_v(e|G)$ be respectively the number of vertices of G lying closer to vertex u than to vertex v and the number of vertices of G lying closer to vertex v than to vertex u, i.e.,

$$n_u(e|G) = \left|\left\{x \in V(G) \,\middle|\, d_G(u,x) < d_G(v,x)\right\}\right|$$

and

$$n_v(e|G) = \left|\left\{x \in V(G) \,\middle|\, d_G(v,x) < d_G(u,x)\right\}\right|.$$

The Szeged index of G, proposed by Gutman [3], is defined as

$$Sz(G) = \sum_{e=uv \in E(G)} n_u(e|G) n_v(e|G).$$

If G is a tree, then $Sz(G)$ is equal to the Wiener index of G [15]. The Szeged index is a graph invariant that has been extensively studied, see, e.g., [2,5–7,9,12,16,17].

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For $w \in V(G)$ and $e = uv \in E(G)$, the distance between vertex w and edge e is defined as the minimum of the distances between w and the ends of e, i.e., $d_G(w,e) = d(w,e) = \min\{d_G(w,u), d_G(w,v)\}$. For $e = uv \in E(G)$, let

$$m_u(e|G) = \left|\left\{f \in E(G) \,\middle|\, d_G(u,f) < d_G(v,f)\right\}\right|$$

and

$$m_v(e|G) = \left|\left\{f \in E(G) \,\middle|\, d_G(v,f) < d_G(u,f)\right\}\right|.$$

The edge Szeged index of G, proposed by Gutman and Ashrafi [4], is defined as

$$Sz_e(G) = \sum_{e=uv\in E(G)} m_u(e|G) m_v(e|G).$$

Some properties of the edge Szeged index may be found in [1,8,14].

Let $\{G_i\}_{i=1}^d$ be a set of finite pairwise vertex-disjoint connected graphs with $v_i \in V(G_i)$. The bridge graph $B(G_1, G_2, \ldots, G_d) = B(G_1, G_2, \ldots, G_d; v_1, v_2, \ldots, v_d)$ of $\{G_i\}_{i=1}^d$ with respect to the vertices $\{v_i\}_{i=1}^d$ is the graph obtained from the graphs $G_1, G_2, \ldots, G_d$ by connecting the vertices v_i and v_{i+1} by an edge for all $i = 1, 2, \ldots, d-1$.

In [10], Mansour and Schork introduced formulas for the vertex PI index and Szeged index of bridge graphs, and used the formulas to compute the vertex PI indices and Szeged indices of several graphs. Work on other graph invariants of bridge graphs may be found in [11]. In this note, we use the techniques from [10] to find a formula for the edge Szeged index of bridge graphs, and by using this formula, the edge Szeged indices of several graphs are computed.

2. Main results

For a graph G with $v \in V(G)$, let $Q_v(G)$ be the set of edges xy of G such that $d_G(v,x) = d_G(v,y)$.

In a bridge graph $B(G_1, G_2, \ldots, G_d)$, for $i = 1, 2, \ldots, d$, let $L(G_i)$ be the set of edges $e = uv$ in $E(G_i) \setminus Q_{v_i}(G_i)$ such that $d_G(u, v_i) < d_G(v, v_i)$ and $R(G_i)$ the set of edges $e = uv$ in $E(G_i) \setminus Q_{v_i}(G_i)$ such that $d_G(u, v_i) > d_G(v, v_i)$. To make this well defined, we choose an arbitrary direction on the edges of G_i (which is fixed for all the following computations); the results do not depend on the direction chosen.

Theorem 2.1. *The edge Szeged index of bridge graph $G = B(G_1, G_2, \ldots, G_d)$ of $\{G_i\}_{i=1}^d$ with respect to the vertices $\{v_i\}_{i=1}^d$ is given by*

$$Sz_e(G) = \sum_{i=1}^d Sz_e(G_i) + \sum_{i=1}^d \left(\left|E(G)\right| - \left|E(G_i)\right|\right)(l_i + r_i) + \sum_{i=1}^{d-1} \alpha_i \left(\left|E(G)\right| - 1 - \alpha_i\right),$$

where

$$l_i = \sum_{e=uv\in L(G_i)} m_v(e|G_i), \qquad r_i = \sum_{e=uv\in R(G_i)} m_u(e|G_i), \qquad \alpha_i = \sum_{j=1}^i \left|E(G_j)\right| + i - 1$$

for all $i = 1, 2, \ldots, d$.

Proof. Let $G = B(G_1, G_2, \ldots, G_d)$. Note that $E(G_i) = Q_{v_i}(G_i) \cup L(G_i) \cup R(G_i)$ for $i = 1, 2, \ldots, d$. From the definition of the edge Szeged index, we have

$$\begin{aligned} Sz_e(G) = \sum_{e=uv\in E(G)} m_u(e|G) m_v(e|G) &= \sum_{i=1}^d \sum_{e=uv\in Q_{v_i}(G_i)} m_u(e|G) m_v(e|G) + \sum_{i=1}^d \sum_{e=uv\in L(G_i)\cup R(G_i)} m_u(e|G) m_v(e|G) \\ &\quad + \sum_{i=1}^{d-1} m_{v_i}(v_i v_{i+1}|G) m_{v_{i+1}}(v_i v_{i+1}|G). \end{aligned}$$

For $i = 1, 2, \ldots, d$, if $e = uv \in Q_{v_i}(G_i)$, then $d(v_i, u) = d(v_i, v)$, and thus for any $f \in E(G) \setminus E(G_i)$, we have $d(u, f) = d(v, f)$, implying that $m_u(e|G) = m_u(e|G_i)$, $m_v(e|G) = m_v(e|G_i)$, and then

$$\sum_{e=uv\in Q_{v_i}(G_i)} m_u(e|G) m_v(e|G) = \sum_{e=uv\in Q_{v_i}(G_i)} m_u(e|G_i) m_v(e|G_i).$$

For $i = 1, 2, \ldots, d$, if $e = uv \in L(G_i)$, then $d(u, v_i) < d(v, v_i)$, and thus

$$m_u(e|G)m_v(e|G) = \left(m_u(e|G_i) + d - 1 + \sum_{\substack{1 \leqslant j \leqslant d \\ j \neq i}} \left|E(G_j)\right|\right) m_v(e|G_i)$$
$$= m_u(e|G_i)m_v(e|G_i) + \left(\left|E(G)\right| - \left|E(G_i)\right|\right) m_v(e|G_i),$$

and if $e = uv \in R(G_i)$, then $d(v, v_i) < d(u, v_i)$, and thus

$$m_u(e|G)m_v(e|G) = m_u(e|G_i)\left(m_v(e|G_i) + d - 1 + \sum_{\substack{1 \leqslant j \leqslant d \\ j \neq i}} \left|E(G_j)\right|\right)$$
$$= m_u(e|G_i)m_v(e|G_i) + \left(\left|E(G)\right| - \left|E(G_i)\right|\right) m_u(e|G_i).$$

Thus

$$\begin{aligned}
\sum_{e=uv \in L(G_i)\cup R(G_i)} m_u(e|G)m_v(e|G) &= \sum_{e=uv\in L(G_i)} m_u(e|G)m_v(e|G) + \sum_{e=uv\in R(G_i)} m_u(e|G)m_v(e|G) \\
&= \sum_{e=uv\in L(G_i)} \left[m_u(e|G_i)m_v(e|G_i) + \left(\left|E(G)\right| - \left|E(G_i)\right|\right) m_v(e|G_i)\right] \\
&\quad + \sum_{e=uv\in R(G_i)} \left[m_u(e|G_i)m_v(e|G_i) + \left(\left|E(G)\right| - \left|E(G_i)\right|\right) m_u(e|G_i)\right] \\
&= \sum_{e=uv\in L(G_i)\cup R(G_i)} m_u(e|G_i)m_v(e|G_i) \\
&\quad + \left(\left|E(G)\right| - \left|E(G_i)\right|\right)\left(\sum_{e=uv\in L(G_i)} m_v(e|G_i) + \sum_{e=uv\in R(G_i)} m_u(e|G_i)\right).
\end{aligned}$$

For $i = 1, 2, \ldots, d-1$, obviously,

$$m_{v_i}(v_i v_{i+1}|G) m_{v_{i+1}}(v_i v_{i+1}|G) = \left(\sum_{j=1}^{i} \left|E(G_j)\right| + i - 1\right)\left[\sum_{j=i+1}^{d} \left|E(G_j)\right| + d - (i+1)\right].$$

Now it follows that

$$\begin{aligned}
Sz_e(G) &= \sum_{i=1}^{d} \sum_{e=uv\in Q_{v_i}(G_i)} m_u(e|G_i)m_v(e|G_i) + \sum_{i=1}^{d} \sum_{e=uv\in L(G_i)\cup R(G_i)} m_u(e|G_i)m_v(e|G_i) + \sum_{i=1}^{d}\left(\left|E(G)\right| - \left|E(G_i)\right|\right) \\
&\quad \times \left(\sum_{e=uv\in L(G_i)} m_v(e|G_i) + \sum_{e=uv\in R(G_i)} m_u(e|G_i)\right) + \sum_{i=1}^{d-1}\left(\sum_{j=1}^{i}\left|E(G_j)\right| + i - 1\right)\left(\sum_{j=i+1}^{d}\left|E(G_j)\right| + d - i - 1\right) \\
&= \sum_{i=1}^{d} \sum_{e=uv\in E(G_i)} m_u(e|G_i)m_v(e|G_i) + \sum_{i=1}^{d}\left(\left|E(G)\right| - \left|E(G_i)\right|\right)(l_i + r_i) + \sum_{i=1}^{d-1} \alpha_i\left(\left|E(G)\right| - 1 - \alpha_i\right),
\end{aligned}$$

as desired. □

For a connected graph H with vertex v, let $G_d(H, v) = B(\underbrace{H, \ldots, H}_{d \text{ times}}; \underbrace{v, \ldots, v}_{d \text{ times}})$. Obviously, $G_1(H, v) = H$. As a corollary of Theorem 2.1, we have

Corollary 2.1. *Let H be a connected graph with m edges, and let $v \in V(H)$. Then*

$$Sz_e\left(G_d(H, v)\right) = d\,Sz_e(H) + (m+1)d(d-1)(l_H + r_H) + (d-1)\left(\frac{(m+1)d(md+m+d-5)}{6} + 1\right),$$

where $l_H = \sum_{e=xy\in L(H)} m_y(e|H)$ and $r_H = \sum_{e=xy\in R(H)} m_x(e|H)$.

Proof. Setting $G_i = H$ and $v_i = v$ with $i = 1, 2, \ldots, d$ in Theorem 2.1, we have $|E(G_d(H, v))| = d|E(H)| + d - 1 = md + d - 1$, and then

$$Sz_e\big(G_d(H,v)\big) = \sum_{i=1}^{d} Sz_e(H) + \sum_{i=1}^{d}\big(\big|E\big(G_d(H,v)\big)\big| - \big|E(H)\big|\big)(l_H + r_H) + \sum_{i=1}^{d-1}\alpha_i\big(\big|E\big(G_d(H,v)\big)\big| - 1 - \alpha_i\big)$$
$$= d\,Sz_e(H) + (m+1)d(d-1)(l_H + r_H) + \sum_{i=1}^{d-1}\alpha_i\big(\big|E\big(G_d(H,v)\big)\big| - 1 - \alpha_i\big),$$

where $\alpha_i = \sum_{j=1}^{i}|E(H)| + i - 1 = (m+1)i - 1$. Note that

$$\sum_{i=1}^{d-1}\alpha_i\big(\big|E\big(G_d(H,v)\big)\big| - 1 - \alpha_i\big) = \sum_{i=1}^{d-1}\big[(m+1)i - 1\big]\big[(m+1)d - (m+1)i - 1\big]$$
$$= \sum_{i=1}^{d-1}\big[-(m+1)^2 i^2 + (m+1)^2 di - (m+1)d + 1\big]$$
$$= -\frac{(m+1)^2 d(d-1)(2d-1)}{6} + \frac{(m+1)^2 d^2(d-1)}{2} - \big[(m+1)d - 1\big](d-1).$$

The result follows easily. □

Let $P_n = u_1u_2\ldots u_n$ be the path on n vertices. Obviously, $P_n = G_n(P_1, u_1)$. By Corollary 2.1,

$$Sz_e(P_n) = (n-1)\left(\frac{n(n-5)}{6} + 1\right) = \binom{n-1}{3}.$$

Let $A_{d,n} = G_d(P_n, u_1)$. Obviously, $A_{d,1} = P_d$ and $A_{1,n} = P_n$.

Corollary 2.2. *The edge Szeged index of $A_{d,n}$ is given by*

$$Sz_e(A_{d,n}) = d\binom{n-1}{3} + 6\binom{d}{2}\binom{n}{3} + n^2\binom{d+1}{3} - (nd-1)(d-1).$$

Proof. By Corollary 2.1, we have

$$Sz_e(A_{d,n}) = d\,Sz_e(P_n) + nd(d-1)(l_{P_n} + r_{P_n}) + (d-1)\left(\frac{nd[(n-1)d + (n-1) + d - 5]}{6} + 1\right)$$
$$= d\,Sz_e(P_n) + nd(d-1)(l_{P_n} + r_{P_n}) + n^2\cdot\frac{(d+1)d(d-1)}{6} - (nd-1)(d-1).$$

Note that $Sz_e(P_n) = \binom{n-1}{3}$ and $l_{P_n} + r_{P_n} = \sum_{i=1}^{n-2} i = \binom{n-1}{2}$. Then the result follows easily. □

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