Combinatorics

Some notes on domination edge critical graphs

Quelques remarques sur les graphes à domination critique par addition d’arête

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Abstract

A graph $G$ is domination edge critical, or just $\gamma$-edge critical, if for any edge $e$ not in $G$, $\gamma(G + e) < \gamma(G)$. We will characterize all connected $\gamma$-edge critical cactus graphs.

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Résumé

Un graphe $G$ est un graphe à domination critique par addition d’arête, ou simplement $\gamma$-critique par arête, si pour toute arête $e$ qui n'est pas dans $G$ on a $\gamma(G + e) < \gamma(G)$. Nous caractérisons les graphes cactus, connexes et $\gamma$-critiques par arête.

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1. Introduction

For notation and terminology in general we follow [4]. Let $G = (V, E) = (V(G), E(G))$ be a graph without isolated vertices. The (open) neighborhood $N(v)$ of a vertex $v \in V$ is the set of vertices which are adjacent to $v$. For a subset $S$, $N(S) = \bigcup_{v \in S} N(v)$, and $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of $G$ if $N[S] = V(G)$. The minimum cardinality of a dominating set of $G$ is the domination number and denoted by $\gamma(G)$. We refer a dominating set of cardinality $\gamma(G)$ as a $\gamma(G)$-set. For references on domination see for example [4].

The concept of domination critical graphs in sense of edge addition is introduced by Sumner and Blitch [5], and further studied by several authors. A graph $G$ is domination edge critical, or just $\gamma$-edge critical, if for any edge $e$ not in $G$, $\gamma(G + e) < \gamma(G)$. It is easy to see that in a $\gamma$-edge critical graph $G$ for any edge not in $G$, $\gamma(G + e) = \gamma(G) - 1$. A graph $G$ is $k$-$\gamma$-edge critical if $G$ is $\gamma$-edge critical and $\gamma(G) = k$. For references on domination edge critical graphs see for example [1–3,5,6].

Several authors studied $\gamma$-edge critical graphs. Sumner and Blitch [5] characterized $\gamma$-edge critical graphs with $\gamma(G) = 1, 2$. They also gave six $\gamma$-edge critical graphs of order $n \leq 8$ with $\gamma(G) = 3$. For $\gamma(G) \geq 3$, characterizing $\gamma$-edge critical graphs is difficult and is still open. Sumner [6] characterized disconnected $\gamma$-edge critical graphs with $\gamma(G) = 3$. Favaron et al. [3] studied the diameter of $\gamma$-edge critical graphs. Ananchuen and Plummer [1,2] studied properties of $\gamma$-edge critical graphs with $\gamma(G) = 3$.

A graph $G$ is called a cactus graph if each edge of $G$ is contained in at most one cycle. A cactus graph having one cycle is called a unicyclic graph and a connected cactus graph with no cycle is called a tree. In this paper we study $\gamma$-edge critical graphs, and characterize all $\gamma$-edge critical cactus graphs.

Let $S$ be a dominating set in a graph $G$ and let $v \in S$. A vertex $w \in V(G)$ is an $S$-private neighbor of $v$ if $N[w] \cap S = \{v\}$. Further, the $S$-private neighborhood of $v$, denoted $pn(v, S)$, is the set of all $S$-private neighbors of $v$. Thus if $pn[v, S] = \{v\}$ then $S - \{v\}$ is a dominating set for $G - v$. We recall that a pendant vertex (or a leaf) is a vertex of degree one, and a
support vertex is a vertex which is adjacent to a pendant vertex. We also call an edge an pendant edge if at least one of its end-points is a pendant vertex.

We make use of the following lemma:

Lemma 1. (See Sumner and Blitch [5].) A graph G with \( \gamma(G) = 1 \) is \( \gamma \)-edge critical if and only if G is \( K_n \).

2. Preliminary results

We first give a characterization of all \( \gamma \)-edge critical graphs.

Theorem 2. A graph G is \( \gamma \)-edge critical if and only if for any two non-adjacent vertices \( x, y \) there is a \( \gamma(G) \)-set \( S \) containing \( x, y \) such that \( pn[x, S] = \{x\} \) or \( pn[y, S] = \{y\} \).

Proof. Let \( G \) be a \( \gamma \)-edge critical graph and \( x, y \) be two non-adjacent vertices of \( G \). Let \( S \) be a \( \gamma(G + xy) \)-set. If \( \{x, y\} \subseteq S \) or \( \{x, y\} \cap S = \emptyset \), then \( S \) is a dominating set for \( G \), a contradiction. Thus assume, without loss of generality, that \( x \in S \) and \( y \notin S \). Now \( D = S \cup \{y\} \) is a dominating set for \( G \), and \( pn[y, D] = \{y\} \). Since \( |S| = \gamma(G) - 1 \), we obtain that \( D \) is a \( \gamma(G) \)-set.

Conversely, let \( x, y \) be two non-adjacent vertices of \( G \). By assumption there is a \( \gamma(G) \)-set \( S \) containing \( x, y \) such that \( \gamma(S) = \{|x| \} \) or \( \gamma(S) = \{|y| \} \). Assume that \( \gamma(S) = \{|x| \} \). Then \( S - \{x\} \) is a dominating set for \( G + xy \). We conclude that \( G \) is \( \gamma \)-edge critical. \( \Box \)

In the following observation we characterize \( \gamma \)-edge critical paths and cycles:

Observation 3.

1. A path \( P_n \) is \( \gamma \)-edge critical if and only if \( n = 2 \).
2. A cycle \( C_n \) is \( \gamma \)-edge critical if and only if \( n = 3 \) or \( 4 \).

To characterizing \( \gamma \)-edge critical trees, we need the following lemmas:

Lemma 4. Any two support vertices in a \( \gamma \)-edge critical graph are adjacent.

Proof. Let \( G \) be a \( \gamma \)-edge critical and \( x, y \) be two support vertices of \( G \). Assume that \( x \) is not adjacent to \( y \). By Theorem 2, there is a \( \gamma(G) \)-set \( S \) containing \( x, y \) such that \( \gamma(S) = \{|x| \} \) or \( \gamma(S) = \{|y| \} \). Assume that \( \gamma(S) = \{|x| \} \). Then any leaf adjacent to \( x \) belongs to \( S \). Now \( S - \{w\} \) is a dominating set for \( G \), where \( w \) is a leaf adjacent to \( x \) which belongs to \( S \). This is a contradiction. \( \Box \)

Similarly the following is verified:

Lemma 5. Any support vertex in a \( \gamma \)-edge critical graph is adjacent to exactly one leaf.

We next characterize all \( \gamma \)-edge critical trees.

Theorem 6. A tree \( T \) is \( \gamma \)-edge critical if and only if \( T = P_2 \).

Proof. Let \( T \) be a \( \gamma \)-edge critical tree. By Lemma 4, \( \text{diam}(T) \leq 3 \). By Lemma 5, \( T \) is a path. Now Observation 3 part (1) implies the result. \( \Box \)

3. Main results

In this section we give our main results. We will characterize all connected \( \gamma \)-edge critical cactus graphs. Recall that the corona \( \text{cor}(G) \) of a graph \( G \) is the graph obtained from \( G \) by adding a pendant edge to any vertex of \( G \). We first investigate whether the corona of a graph is \( \gamma \)-edge critical.

Lemma 7. The corona \( \text{cor}(G) \) of a connected graph \( G \) is \( \gamma \)-edge critical if and only if \( G \) is a complete graph with at least three vertices.

Proof. Let \( \text{cor}(G) \) be \( \gamma \)-edge critical. It is obvious that \( \gamma(\text{cor}(G)) = |V(G)| \). If there are two non-adjacent vertices \( x, y \) in \( G \), then \( \gamma(\text{cor}(G + xy)) = |V(G)| = \gamma(\text{cor}(G)) \), a contradiction. Thus \( G \) is a complete graph. Assume that \( |V(G)| = 2 \). Then \( \text{cor}(G) \) is the path \( P_4 \), and so \( \gamma(\text{cor}(G)) = 2 \). If \( x, y \) are the two end-points of \( \text{cor}(G) \), then \( \gamma(\text{cor}(G + xy)) = \gamma(C_4) = 2 \). This is a contradiction, since \( G \) is \( \gamma \)-edge critical. Thus \( |V(G)| \geq 3 \).
Conversely let $G$ be the complete graph with at least three vertices. Let $x, y$ be two leaves of $\text{cor}(G)$ and $x_1, y_1$ be the support vertices adjacent to $x, y$, respectively. It follows that $(V(G) - \{x_1, y_1\}) \cup \{x, y\}$ is a dominating set for $\text{cor}(G) + xy$, and $V(G) - \{y_1\}$ is a dominating set for $\text{cor}(G) + x_1 y$. Since $x, y$ have been chosen arbitrarily, the result follows. □

**Lemma 8.** If $G$ is a graph with a path $v_1 - v_2 - v_3 - v_4$ such that $v_1 \notin N(v_4)$ and $\deg(v_i) = 2$ for $i = 2, 3$, then $G$ is not $\gamma$-edge critical.

**Proof.** Let $G$ be a graph with a path $v_1 - v_2 - v_3 - v_4$ such that $\deg(v_i) = 2$ for $i = 2, 3$. Assume that $G$ is $\gamma$-edge critical. By Theorem 2, there is a $\gamma(G)$-set $S$ containing $v_1, v_4$ such that $pn[v_1, S] = \{v_1\}$ or $pn[v_4, S] = \{v_4\}$. Without loss of generality assume that $pn[v_4, S] = \{v_4\}$. Then $S \cap \{v_2, v_3\} = \emptyset$. Now $S \setminus \{v_2, v_3\}$ is a dominating set for $G$, a contradiction. □

**Lemma 9.** Let $x$ be a leaf and $C$ be a cycle in a connected graph $G$ such that $d(x, C) \geq 2$ and every vertex of $C$ except one is of degree two, then $G$ is not $\gamma$-edge critical.

**Proof.** Let $x$ be a leaf and $C$ be a cycle in a graph $G$ such that $d(x, C) \geq 2$ and every vertex of $C$ except one is of degree two. Assume that $G$ is $\gamma$-edge critical. Let $z \in V(C)$ be a vertex with $d(x, z) = d(x, C) = d$, and let $P$ be a shortest path between $x$ and $z$. Let $b \in N(z)$ be on $P$. By Lemma 8, $|V(C)| \leq 4$. Let $w \in N(z) \cap V(C)$. By Theorem 2, there is a $\gamma(G)$-set $S$ containing $w, b$ such that $pn[w, S] = \{w\}$ or $pn[b, S] = \{b\}$. If $pn[w, S] = \{w\}$, then $|V(C) \cap S| \geq 2$, which implies that $|V(C)| = 4$. Now $(S \setminus V(C)) \cup \{v\}$ is a dominating set for $G$, where $v \in V(C) - N[z]$. This is a contradiction. Thus we may assume that $pn[b, S] = \{b\}$. Now $(S - (V(C) \cup \{b\})) \cup (V(C) - N[z]) \cup \{z\}$ is a dominating set for $G$. This is a contradiction. □

We are now ready to characterize all unicyclic $\gamma$-edge critical graphs.

**Theorem 10.** A connected unicyclic graph $G$ is $\gamma$-edge critical if and only if $G$ is $C_3, C_4$, or $\text{cor}(C_3)$.

**Proof.** First it is easy to see that $C_3, C_4$, and $\text{cor}(C_3)$ are $\gamma$-edge critical. Let $G$ be a unicyclic $\gamma$-edge critical graph, and let $C$ be the unique cycle of $G$. If $G = C$, then by Observation 3 part (2), $G \in \{C_3, C_4\}$. So we assume that $G \neq C$. Let $x$ be a leaf of $G$ such that $d(x, C)$ is maximum, and let $y \in V(C)$ be the vertex with $d(x, y) = d(x, C)$. By Lemmas 4 and 9, $d(x, C) \leq 1$ and so $d(x, C) = 1$. By Lemma 5, $\deg(y) = 3$. We show that $G = \text{cor}(C)$. Assume that $G \neq \text{cor}(C)$. Then we assume that $C$ has some vertex of degree 2. By Lemmas 4 and 8, $|V(C)| \leq 4$.

If $|V(C)| = 4$, then at most one vertex in $N(y)$ is a support vertex. If there is no support vertex in $N(y)$, then it is easy to see that $G$ is not $\gamma$-edge critical. We may now assume that there is a support vertex $a \in N(y) \cap V(C)$. Let $a_1$ be the leaf adjacent to $a$. Then $\gamma(G) = \gamma(G + a_1 x) = 2$, a contradiction.

Thus we assume that $|V(C)| = 3$. Let $V(C) = \{y, a, b\}$. Since $G$ has some vertex of degree 2, we assume that $\deg(b) = 2$. If $\deg(a) = 2$, then $\gamma(G) = 1$ and by Lemma 1, $G$ is not $\gamma$-edge critical. So assume that $\deg(a) = 3$. Let $a_1$ be the leaf adjacent to $a$. Then $\gamma(G) = \gamma(G + a_1 x) = 2$, a contradiction.

We conclude that $G = \text{cor}(C)$. Then by Lemma 7, $|V(C)| = 3$ and so $G = \text{cor}(C_3)$. □

Our next aim is to characterize all $\gamma$-edge critical cactus graphs with at least two cycles.

**Lemma 11.** If $G$ is a $\gamma$-edge critical cactus graph with at least two cycles, then $\delta(G) \geq 2$.

**Proof.** Let $G$ be a $\gamma$-edge critical cactus graph with $k \geq 2$ cycles. Let $C_1, C_2, \ldots, C_k$ be the cycles of $G$. Assume that $\delta(G) = 1$. Let $x$ be a leaf of $G$. Without loss of generality assume that $d(x, C_1) \leq d(x, C_2) \leq d(x, C_3) \leq \cdots \leq d(x, C_k)$ for $i = 1, 2, \ldots, k$. Let $z \in V(C_2)$ be the vertex with $d(x, z) = d(x, C_2) = d$, and let $P$ be the shortest path between $x$ and $z$. If $d(x, z) \geq 2$, then by Lemma 4, any vertex of $V(C_2) - \{z\}$ is of degree two, and by Lemma 9, $G$ is not $\gamma$-edge critical which is a contradiction. Thus $d(x, z) \leq 1$.

Suppose next that $d(x, z) = 1$. Thus $d(x, C_i) = 1$ for $i = 1, 2, \ldots, k$, and $V(C_1) \cap V(C_2) \cap \cdots \cap V(C_k) = \{z\}$. By Lemma 5, $x$ is the only leaf adjacent to $z$. Let $w_1 \in N(z) \cap V(C_1)$ and $w_2 \in N(z) \cap V(C_2)$. By Theorem 2, there is a $\gamma(G)$-set $S$ containing $w_1, w_2$ such that $pn[w_1, S] = \{w_1\}$ or $pn[w_2, S] = \{w_2\}$. But then $(S - \{w_1, w_2, x\}) \cup \{z\}$ is a dominating set for $G$, a contradiction. We deduce that $d(x, y) = 0$, contradicting that $x$ is a leaf. □

**Theorem 12.** There is no $\gamma$-edge critical cactus graph with at least two cycles.

**Proof.** Assume to the contrary that $G$ is a $\gamma$-edge critical cactus graph with at least two cycles. Let $C_1, C_2, \ldots, C_k$ be the cycles of $G$. By Lemma 11, $\delta(G) \geq 2$. Without loss of generality assume that $d(C_1, C_2) \leq d(C_i, C_j)$ for $1 \leq i, j \leq k$ and $i \neq j$. By Lemma 8, $|V(C_i)| \leq 4$ for $i = 1, 2$. Let $x \in V(C_1)$ and $y \in V(C_2)$ be two vertices with $d(x, y) = d(C_1, C_2)$.

We show that $d(x, y) = 0$. Assume that $d(x, y) \geq 1$. Let $a \in N(x) \cap V(C_1)$ and $b \in N(y) \cap V(P)$. By Theorem 2, there is a $\gamma(G)$-set $S$ containing $a, b$ such that $pn[a, S] = \{a\}$ or $pn[b, S] = \{b\}$. Suppose that $pn[b, S] = \{b\}$. Then $S - (V(C_1) \cup \{b\}) \cup \{x\} \cup (V(C_1) - N[x])$ is a dominating set for $G$, a contradiction. Thus $pn[a, S] = \{a\}$. Then $|S \cap V(C_1)| \geq 2$. This implies that $|V(C_1)| = 4$. Now $(S - V(C_2) \cup (V(C_2) - N[x]))$ is a dominating set for $G$, a contradiction. Hence $d(x, y) = 0$. □
This implies that \( V(C_1) \cap V(C_2) \cap \cdots \cap V(C_k) = \{x\} \). Let \( a_1 \in N(x) \cap V(C_1) \) and \( b_1 \in N(x) \cap V(C_2) \). By Theorem 2, there is a \( \gamma(G) \)-set \( S \) containing \( a_1, b_1 \) such that \( pn[a_1, S] = \{a_1\} \) or \( pn[b_1, S] = \{b_1\} \). But \( |V(C_i)| \leq 4 \) for \( i = 1, 2 \). Now it is a routine matter to see that \( G \) is not \( \gamma \)-edge critical, a contradiction. \( \square \)

Now from Theorems 6, 10, and 12 we obtain the following:

**Theorem 13.** A connected cactus graph \( G \) is \( \gamma \)-edge critical if and only if \( G \) is \( P_2, C_3, C_4, \) or \( \text{cor}(C_3) \).

We close with the following problem:

**Problem 14.** Characterize all connected \( \gamma \)-edge critical graphs \( G \) with \( \delta(G) = 1 \).

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**References**