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Partial Differential Equations

Self-similar solutions with fat tails for a coagulation equation with diagonal kernel

Solutions auto-similaires avec queues épaisses d'une équation de coagulation à noyau diagonal

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ABSTRACT

We consider self-similar solutions of Smoluchowski's coagulation equation with a diagonal kernel of homogeneity $\gamma < 1$. We show that there exists a family of second-kind self-similar solutions with power-law behavior $x^{-(1+\rho)}$ as $x \to \infty$ with $\rho \in (\gamma, 1)$. To our knowledge this is the first example of a non-solvable kernel for which the existence of such a family has been established.

RÉSUMÉ

Nous considérons des solutions autosimilaires de l'équation de coagulation de Smoluchowski avec un noyau diagonal d'homogénéité $\gamma < 1$. Nous prouvons l'existence d'une famille de solutions autosimilaires de deuxième type avec comportement à l'infini en puissance $x^{-(1+\rho)}$, $\rho \in (\gamma, 1)$. A notre connaissance, ceci constitue le premier exemple d'existence d'une telle famille pour un noyau non explicitement résoluble.

1. Introduction

Smoluchowski's coagulation equation provides a mean-field description of binary coalescence of clusters. If ξ denotes the size of a cluster and $f(\xi, t)$ the corresponding number density at time t then the equation is

$$\frac{\partial}{\partial t} f(\xi, t) = \frac{1}{2} \int_0^\xi \mathrm{d}\eta\, K(\xi - \eta, \eta) f(\eta, t) f(\xi - \eta, t) - f(\xi, t) \int_0^\infty \mathrm{d}\eta\, K(\xi, \eta) f(\eta, t), \tag{1}$$

where $K(\xi, \eta)$ is a kernel that describes the rate of the coalescence process.

Here we consider a specific diagonal kernel of homogeneity $\gamma < 1$, given by $K(\xi, \eta) = \delta_{(\xi-\eta)} \xi^{1+\gamma}$, that reduces (1) to

$$\frac{\partial}{\partial t} f(\xi, t) = \frac{1}{4} \left(\frac{\xi}{2}\right)^{1+\gamma} f^2\left(\frac{\xi}{2}, t\right) - \xi^{1+\gamma} f^2(\xi, t). \tag{2}$$

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In the following we study self-similar solutions of (2). Such solutions are of the form

$$f(\xi,t)=t^{-(1+(1+\gamma)\beta)}g\left(\frac{\xi}{t^\beta}\right) \tag{3}$$

for some positive β, where g satisfies, with $x=\xi/t^\beta$, that

$$-\big(1+(1+\gamma)\beta\big)g-\beta x g'(x)=\frac{1}{4}\left(\frac{x}{2}\right)^{1+\gamma}g^2\left(\frac{x}{2}\right)-x^{1+\gamma}g^2(x). \tag{4}$$

If one looks for solutions with conserved mass, then β is uniquely determined by $\beta=\beta_*:=1/(1-\gamma)$. For further reference we also note that we can integrate the equation in (4) to obtain

$$\beta x^2 g(x)=\int_{x/2}^{x} s^{2+\gamma}g^2(s)\,\mathrm{d}s+(1-\gamma)(\beta-\beta_*)\int_0^x sg(s)\,\mathrm{d}s. \tag{5}$$

Here we assumed implicitly that $xg(x)$ and $x^{2+\gamma}g^2(x)$ are integrable at zero and that $\lim_{x\to 0}x^2g(x)=0$. As we will see below (cf. (9)), these properties will be satisfied by the solutions we are going to consider. Notice also that we have the well-known power-law solution

$$g=x^{-(1+\gamma)}\frac{1}{1-\theta}\quad \text{with } \theta:=2^{\gamma-1}<1. \tag{6}$$

In [1] a mass-conserving solution of (5), that is a solution for $\beta=\beta_*$, is constructed that is decaying exponentially fast and satisfies

$$g(x)=x^{-(1+\gamma)}\left(\frac{1}{1-\theta}-cx^{\mu/(1-\gamma)}+o\big(x^{\mu/(1-\gamma)}\big)\right)\quad \text{as } x\to 0, \tag{7}$$

where $\mu>0$ satisfies a certain transcendental equation. The constant $c>0$ is not determined due to an invariance of (4) under the rescaling $g(x)\mapsto a^{1+\gamma}g(ax)$ for any $a>0$. In the case of mass-preserving solutions the constant can be fixed by normalizing the mass of the solution. As is pointed out in [1], the solution is unique in the class of functions satisfying (7), but uniqueness in general is not known.

In [1] the question is raised whether solutions with algebraic decay, others from the one in (6), exist in analogy to the ones that have been found in [2] for the constant and additive kernel. More precisely, for example for the constant kernel, it is established in [2] that there exists a family of self-similar solutions with infinite mass and the decay behavior $x^{-(1+\rho)}$ for all $\rho\in(0,1)$. Furthermore, it is shown that a solution of the coagulation equation converges to the self-similar solution with decay behavior $x^{-(1+\rho)}$ if and only if the mass-distribution of the initial data is regularly varying with exponent $1-\rho$. In this note we prove for the diagonal kernel the existence of a corresponding family of self-similar solutions with infinite mass and asymptotic behavior $x^{-(1+\rho)}$ as $x\to\infty$ with $\rho\in(\gamma,1)$. Notice, that this includes solutions that are increasing as $x\to\infty$ if $\gamma<-1$. Our proof is simple and exploits strong monotonicity properties of a suitably rescaled version of the equation for the self-similar solution. We presently do not know, however, how to characterize the domains of attraction of these self-similar solutions. The analysis in [2] relies on the fact that the Laplace transform of the equation satisfies a simple ODE, a method that is not applicable in the present situation.

Our main result is the following:

Theorem 1. *Let $\gamma<1$ and μ be the unique positive solution of*

$$\frac{1+\beta\mu}{2}=\frac{1-2^{\gamma-1-\mu}}{1-2^{\gamma-1}}. \tag{8}$$

Then there exists for any $\beta>\beta_*$ a solution g of (5) such that

$$g(x)=x^{-(1+\gamma)}\left(\frac{1}{1-\theta}-cx^{\mu/(1-\gamma)}+o\big(x^{\mu/(1-\gamma)}\big)\right) \tag{9}$$

as $x\to 0$ with a positive constant c. Furthermore, $x^{-(1+\gamma)}g(x)$ is monotonically decreasing and satisfies

$$g(x)\sim\frac{d}{x^{1+\gamma+1/\beta}}\quad \text{as } x\to\infty \tag{10}$$

for some positive constant d.

As explained above, the constants c and d in Theorem 1 are not determined due to the invariance of the equation under appropriate rescaling.

2. Proof

Our proof proceeds similarly to the one in [1] for the mass-conserving solutions. First, to scale out the singular behavior as $x \to 0$, we introduce $h(x) = g(x)x^{1+\gamma}$ such that h solves

$$-\beta x h'(x) - h(x) = \theta h^2\left(\frac{x}{2}\right) - h^2(x) \tag{11}$$

or, due to (5),

$$\beta x^{1-\gamma} h(x) = \int_{x/2}^{x} s^{-\gamma} h^2(s)\,\mathrm{d}s + (1-\gamma)(\beta-\beta_*)\int_0^x s^{-\gamma} h(s)\,\mathrm{d}s. \tag{12}$$

Notice, that the power-law solution (6) corresponds to the constant solution $h \equiv 1/(1-\theta)$. It is also clear that any solution of (11) for which $\lim_{x\to 0} h(x)$ exists, that this limit must equal $1/(1-\theta)$. We are now looking for solutions that bifurcate from this constant at $x \to 0$.

In order to identify the next order behavior, we make the ansatz $h(x) = 1/(1-\theta) + x^{\mu} + o(x^{\mu})$ as $x \to 0$. Plugging this into (12), recalling that $\beta_* = 1/(1-\gamma)$ and rearranging we find that μ must indeed satisfy (8). If we denote by $F(\mu) = (1-2^{\gamma-1-\mu})/(1-\theta)$ we see that $F(0) = 1 > 1/2$. On the other hand, F is increasing and $\lim_{\mu\to\infty} F(\mu) = 1/(1-\theta)$. Hence, there must be a unique positive solution of (8).

Next, we introduce the function $j(x)$ via

$$h(x) = \frac{1}{1-\theta} + x^{\mu}\bigl(-c + j(x)\bigr), \tag{13}$$

where $c \in \mathbb{R}$ is a constant. Using Eqs. (8) and (12) we obtain that j satisfies

$$\begin{aligned} j(x) = \frac{1}{\beta}x^{-(1-\gamma+\mu)}\Biggl(&\int_{x/2}^{x} s^{-\gamma+\mu}\frac{2}{1-\theta} j(s)\,\mathrm{d}s + \int_{x/2}^{x} s^{-\gamma+2\mu}\bigl(-c+j(s)\bigr)^2\,\mathrm{d}s \\ &+ (1-\gamma)(\beta-\beta_*)\int_0^x s^{-\gamma+\mu} j(s)\,\mathrm{d}s\Biggr) =: T[j]. \end{aligned} \tag{14}$$

In order to prove that a local solution of (14) exists, we can proceed analogously to [1]. We only indicate the main steps here.

We define for some $\varepsilon \in (0,\mu)$ and $z > 0$ the space

$$C_{\varepsilon}(z) := \Bigl\{ f \in C[0,z];\ f(0)=0;\ \|f\| := \sup_{x\in[0,z]} x^{-\varepsilon}\bigl|f(x)\bigr| < \infty \Bigr\}.$$

It is clear that the operator T maps $C_{\varepsilon}(z)$ into itself. Next, we are going to show that T maps a ball in $C_{\varepsilon}(z)$ of a sufficiently small radius R into itself if z is sufficiently small. This follows from

$$\begin{aligned} \bigl\|T[j]\bigr\| \leqslant \frac{1}{\beta}\|j\|\Biggl(&\frac{2}{1-\theta}\frac{1}{1-\gamma+\mu+\varepsilon}\bigl(1-2^{\gamma-1-\mu-\varepsilon}\bigr) + \|j\|\frac{2z^{\mu}}{1-\gamma+2\mu+2\varepsilon}\bigl(1-2^{\gamma-1-\mu-2\varepsilon}\bigr) \\ &+ c^2\frac{2z^{\mu}}{1-\gamma+2\mu} + \frac{(1-\gamma)(\beta-\beta_*)}{1-\gamma+\mu+\varepsilon}\Biggr) \end{aligned}$$

that implies

$$\bigl\|T[j]\bigr\| \leqslant \|j\|\Biggl(\frac{1}{\beta(1-\gamma+\mu+\varepsilon)}\bigl(2F(\mu+\varepsilon) + (1-\gamma)\beta - 1\bigr)\Biggr) + Cz^{\mu}\bigl(\|j\|^2 + 1\bigr).$$

Now we know by the definition of μ that $2F(\mu+\varepsilon) < 1 + \beta(\mu+\varepsilon)$ and hence

$$\frac{1}{\beta(1-\gamma+\mu+\varepsilon)}\bigl(2F(\mu+\varepsilon) + (1-\gamma)\beta - 1\bigr) < \frac{1}{1-\gamma+\mu+\varepsilon}(\mu+\varepsilon+1-\gamma) = 1.$$

Thus, there exists a constant $\kappa = \kappa(\varepsilon) < 1$ such that if $\|j\| \leqslant R$ we find $\|T[j]\| \leqslant \kappa R + Cz^{\mu}(R^2+1)$. For sufficiently small z and an appropriately small R the right-hand side is bounded by R. Similarly one can show that T is a contraction, we omit the details here. Hence, a local solution to (14) exists, and thus also to (11). Next, we choose $c > 0$, and claim that h is decreasing in a neighborhood of zero. To see this, notice that it follows from (14) that $j'(x)$ exists for $x > 0$ and that we have the estimate $|j'(x)| \leqslant C\frac{|j(x)|}{x} + Cx^{\mu-1}$ for $x \in (0,z)$. This in turn implies that

$$h'(x) = \mu x^{\mu-1}\bigl(-c + j(x)\bigr) + x^{\mu} j'(x) \leqslant x^{\mu-1}\bigl(-c\mu + C\mu\bigl|j(x)\bigr| + Cx^{\mu}\bigr).$$

If z is sufficiently small, we find that $h'(x) < 0$ for $x \in (0, z)$. We are going to show that as long as h exists and is positive this property is conserved. Indeed, assume that there exists $x_0 > 0$ such that $h'(x_0) = 0$. Then (11) and the fact that h is decreasing for $x < x_0$ imply that

$$0 = h(x_0)^2 - h(x_0) - \theta h^2\left(\frac{x_0}{2}\right)^2 < (1-\theta)h^2(x_0) - h(x_0) = h(x_0)\bigl((1-\theta)h(x_0) - 1\bigr).$$

As long as h is positive, the right-hand side is strictly negative, since $h(x_0) < 1/(1-\theta)$ and we obtain the desired contradiction. Moreover, Eq. (12) implies for $\beta \geqslant \beta_*$ that h is positive whenever it exists. Hence, using standard results on ordinary differential equations, we obtain global existence of a solution h to (11) which is strictly decreasing. Since Eq. (11) has the only stationary points $1/(1-\theta)$ and 0, it also follows that $h(x) \to 0$ as $x \to \infty$.

It remains to show that $h(x) \sim dx^{-1/\beta}$ as $x \to \infty$ from which (10) follows. First, due to the invariance of Eq. (11) under the transformation $x \to ax$ for $a > 0$, we can assume without loss of generality that $h(1) = 1/2$. Since h satisfies $\beta x h'(x) + h(x) \leqslant h^2(x)$ we have by simple comparison that

$$h(x) \leqslant \frac{1}{1 + x^{1/\beta}} \quad \text{for } x \geqslant 1. \tag{15}$$

We now introduce $p(x) = x^{1/\beta} h(x)$ that solves

$$\beta p'(x) = x^{-(1+1/\beta)}\left(p^2(x) - \theta 2^{2/\beta} p^2\left(\frac{x}{2}\right)\right). \tag{16}$$

The estimate (15) in particular implies that $p(x) \leqslant 1$ for all $x \geqslant 1$ and thus (16) implies that $\beta|p'(x)| \leqslant 2x^{-(1+1/\beta)}$ for all $x \geqslant 2$. Hence $|p(x) - p(x_0)| \leqslant 2x_0^{-1/\beta}$ for any $x_0 \geqslant 2$ which implies that $\lim_{x\to\infty} p(x)$ exists. In order to complete the proof of Theorem 1 it remains to establish that this limit is strictly positive. To this end we note that (12) implies

$$\beta x^{1-\gamma} h(x) > (1-\gamma)(\beta - \beta_*) \int_0^x s^{-\gamma} h(s)\, ds. \tag{17}$$

If we define $\Phi(x) := \int_0^x s^{-\gamma} h(s)\, ds$ then (17) implies that $\beta x \Phi'(x) - (1-\gamma)(\beta - \beta_*)\Phi(x) > 0$. Integrating this last inequality we obtain $(x^{-\frac{(1-\gamma)(\beta-\beta_*)}{\beta}} \Phi(x))' > 0$ and thus

$$x^{-\frac{(1-\gamma)(\beta-\beta_*)}{\beta}} \Phi(x) \geqslant \Phi(1) = \int_0^1 s^{-\gamma} h(s)\, dx =: c_0 > 0$$

for all $x \geqslant 1$. Thus

$$\Phi(x) \geqslant c_0 x^{\frac{(1-\gamma)(\beta-\beta_*)}{\beta}} = c_0 x^{1-\gamma} x^{-\frac{1}{\beta}}$$

for $x \geqslant 1$ and plugging this into (17) we find $h(x) \geqslant \frac{c_0}{\beta} x^{-\frac{1}{\beta}}$ for all $x \geqslant 1$, that finishes the proof.

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