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Mathematical Problems in Mechanics

Legendre–Fenchel duality in elasticity

Dualité de Legendre–Fenchel en élasticité

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ABSTRACT

We show that the displacement and strain formulations of the displacement–traction problem of three-dimensional linearized elasticity can be viewed as Legendre–Fenchel dual problems to the stress formulation of the same problem. We also show that each corresponding Lagrangian has a saddle-point, thus fully justifying this new approach to elasticity by means of Legendre–Fenchel duality.

RÉSUMÉ

On montre que les formulations en déplacements et en déformations du problème de l'élasticité linéarisée tri-dimensionnelle avec des conditions aux limites mixtes peuvent être vues comme des problèmes duaux de Legendre–Fenchel de la formulation en contraintes de ce même problème. On montre également que chacun des Lagrangiens correspondants a un point-selle, justifiant ainsi complètement cette nouvelle approche de l'élasticité au moyen de la dualité de Legendre–Fenchel.

1. Legendre–Fenchel duality

All vector spaces, matrices, etc., considered in this Note are real. The dual space of a normed vector space X is denoted by X^*, and ${}_{X^*}\langle\cdot,\cdot\rangle_X$ designates the associated duality. The bidual space of X is denoted by X^{**}; if X is a reflexive Banach space, X^{**} will be identified with X by means of the usual canonical isometry. The *indicator function* I_A of a subset A of a set X is the function I_A defined by $I_A(x) := 0$ if $x \in A$ and $I_A(x) := +\infty$ if $x \notin A$. A function $g : X \to \mathbb{R} \cup \{+\infty\}$ is *proper* if $\{x \in X;\ g(x) < +\infty\} \neq \varnothing$.

Let Σ be a normed vector space and let $g : \Sigma \to \mathbb{R} \cup \{+\infty\}$ be a proper function. The *Legendre–Fenchel transform* of g is the function $g^* : \Sigma^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$g^* : e \in \Sigma^* \to g^*(e) := \sup_{\sigma\in\Sigma}\bigl\{{}_{\Sigma^*}\langle e, \sigma\rangle_\Sigma - g(\sigma)\bigr\}.$$

The next theorem summarizes some basic properties of the Legendre–Fenchel transform when the space Σ is a reflexive Banach space. For proofs, see, e.g., Ekeland and Temam [7] or Brezis [2]. The equality $g^{**} = g$ constitutes the *Fenchel–Moreau theorem*.

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Theorem 1.1. *Let Σ be a reflexive Banach space, and let $g : \Sigma \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semi-continuous function. Then the Legendre–Fenchel transform $g^* : \Sigma^* \to \mathbb{R} \cup \{+\infty\}$ of g is also proper, convex, and lower semi-continuous. Let*

$$g^{**} : \sigma \in \Sigma^{**} \to g^{**}(\sigma) := \sup_{e \in \Sigma^*} \{ {}_{\Sigma^*}\langle e, \sigma \rangle_\Sigma - g^*(e) \}$$

denote the Legendre–Fenchel transform of g^* (recall that X^{**} is here identified with X). *Then $g^{**} = g$.*

Given a *minimization problem* $\inf_{\sigma \in \Sigma} G(\sigma)$, called $(\mathcal{P})$, with a function $G : \Sigma \to \mathbb{R} \cup \{+\infty\}$ of the specific form given in Theorem 1.2 below, the following simple result will be the basis for defining *two different dual problems of problem* $(\mathcal{P})$. The functions $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ defined in the next theorem are the *Lagrangians* associated with the minimization problem $(\mathcal{P})$.

Theorem 1.2. *Let Σ and V be two reflexive Banach spaces, let $g : \Sigma \to \mathbb{R} \cup \{+\infty\}$ and $h : V^* \to \mathbb{R} \cup \{+\infty\}$ be two proper, convex, and lower semi-continuous functions, let $\Lambda : \Sigma \to V^*$ be a linear and continuous mapping, let the function $G : \Sigma \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$G : \sigma \in \Sigma \to G(\sigma) := g(\sigma) + h(\Lambda\sigma),$$

and finally, let the two functions

$$\mathcal{L} : \Sigma \times \Sigma^* \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\} \quad \text{and} \quad \widetilde{\mathcal{L}} : \Sigma \times V \to \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$$

be defined by

$$\mathcal{L} : (\sigma, e) \in \Sigma \times \Sigma^* \to \mathcal{L}(\sigma, e) := {}_{\Sigma^*}\langle e, \sigma \rangle_\Sigma - g^*(e) + h(\Lambda\sigma),$$
$$\widetilde{\mathcal{L}} : (\sigma, v) \in \Sigma \times V \to \widetilde{\mathcal{L}}(\sigma, v) := g(\sigma) + {}_{V^*}\langle \Lambda\sigma, v \rangle_V - h^*(v).$$

Then

$$\inf_{\sigma \in \Sigma} G(\sigma) = \inf_{\sigma \in \Sigma} \sup_{e \in \Sigma^*} \mathcal{L}(\sigma, e) = \inf_{\sigma \in \Sigma} \sup_{v \in V} \widetilde{\mathcal{L}}(\sigma, v).$$

A key issue then consists in deciding whether the infimum found in problem $(\mathcal{P})$ is equal to the supremum found in either one of its dual problems, i.e., for instance in the case of the first dual problem (to fix ideas), whether

$$\inf_{\sigma \in \Sigma} G(\sigma) = \sup_{e \in \Sigma^*} G^*(e), \quad \text{or equivalently,} \quad \inf_{\sigma \in \Sigma} \sup_{e \in \Sigma^*} \mathcal{L}(\sigma, e) = \sup_{e \in \Sigma^*} \inf_{\sigma \in \Sigma} \mathcal{L}(\sigma, e).$$

If this is the case, the next issue consists in deciding whether the Lagrangian $\mathcal{L}$ possesses a *saddle-point* $(\bar{\sigma}, \bar{e}) \in \Sigma \times \Sigma^*$, i.e., that satisfies

$$\inf_{\sigma \in \Sigma} \sup_{e \in \Sigma^*} \mathcal{L}(\sigma, e) = \inf_{\sigma \in \Sigma} \mathcal{L}(\sigma, \bar{e}) = \mathcal{L}(\bar{\sigma}, \bar{e}) = \sup_{e \in \Sigma^*} \mathcal{L}(\bar{\sigma}, e) = \sup_{e \in \Sigma^*} \inf_{\sigma \in \Sigma} \mathcal{L}(\sigma, e).$$

This is precisely the type of questions addressed in this Note, the point of departure $(\mathcal{P})$ being a classical quadratic minimization problem arising in three-dimensional linearized elasticity.

2. Functional analytic preliminaries

Latin indices vary in the set $\{1, 2, 3\}$, save when they are used for indexing sequences, and the summation convention with respect to repeated indices is systematically used in conjunction with this rule.

A *domain in* $\mathbb{R}^3$ is a bounded, connected, open subset of $\mathbb{R}^3$ whose boundary, denoted by Γ, is Lipschitz-continuous, the set Ω being locally on a single side of Γ.

Spaces of functions, vector fields in $\mathbb{R}^3$, and 3×3 *symmetric* matrix fields, defined over an open subset of $\mathbb{R}^3$ are respectively denoted by italic capitals, boldface Roman capitals, and special Roman capitals. The inner-product of $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$ is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$. The notation $\boldsymbol{s} : \boldsymbol{t} := s_{ij} t_{ij}$ designates the matrix inner-product of two matrices $\boldsymbol{s} := (s_{ij})$ and $\boldsymbol{t} := (t_{ij})$ of order three.

The inner-product in the space $\mathbb{L}^2(\Omega)$ is given by $(\boldsymbol{\sigma}, \boldsymbol{\tau}) \in \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega) \to \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \, dx$, and $\| \cdot \|_{\mathbb{L}^2(\Omega)}$ denotes the corresponding norm. The space $\mathbb{L}^2(\Omega)$ will be identified with its dual space. The duality bracket between the space $\boldsymbol{H}^{1/2}(\Gamma)$ and its dual space will be denoted by $\langle \cdot, \cdot \rangle_\Gamma := {}_{\boldsymbol{H}^{-1/2}(\Gamma)}\langle \cdot, \cdot \rangle_{\boldsymbol{H}^{1/2}(\Gamma)}$.

For any vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{D}'(\Omega)$, the associated *linearized strain tensor* is the symmetric matrix field $\nabla_s \boldsymbol{v} \in \mathbb{D}'(\Omega)$ defined by $\nabla_s \boldsymbol{v} := \frac{1}{2}(\nabla \boldsymbol{v}^T + \nabla \boldsymbol{v})$.

We now recall some functional analytic preliminaries, due to Geymonat and Suquet [10] and Geymonat and Krasucki [8,9]. Given a domain Ω in $\mathbb{R}^3$, define the space

$$\mathbb{H}(\mathbf{div};\Omega) := \{\boldsymbol{\mu} \in \mathbb{L}^2(\Omega);\ \mathbf{div}\,\boldsymbol{\mu} \in \boldsymbol{L}^2(\Omega)\}.$$

The set Ω being a domain, the density of the space $\mathbb{C}^\infty(\overline{\Omega})$ in the space $\mathbb{H}(\mathbf{div};\Omega)$ then implies that the mapping $\boldsymbol{\mu} \in \mathbb{C}^\infty(\overline{\Omega}) \to \boldsymbol{\mu}\boldsymbol{\nu}|_\Gamma$ can be extended to a continuous linear mapping from the space $\mathbb{H}(\mathbf{div};\Omega)$ into $\boldsymbol{H}^{-1/2}(\Gamma)$, which for convenience will be simply denoted by $\boldsymbol{\mu} \in \mathbb{H}(\mathbf{div};\Omega) \to \boldsymbol{\mu}\boldsymbol{\nu} \in \boldsymbol{H}^{-1/2}(\Gamma)$.

Theorem 2.1. *The Green formula*

$$\int_\Omega \boldsymbol{\mu} : \nabla_s \boldsymbol{v}\,\mathrm{d}x + \int_\Omega (\mathbf{div}\,\boldsymbol{\mu}) \cdot \boldsymbol{v}\,\mathrm{d}x = \langle \boldsymbol{\mu}\boldsymbol{\nu}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma$$

holds for all $\boldsymbol{\mu} \in \mathbb{H}(\mathbf{div};\Omega)$ and all $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$.

The following extension of the classical *Donati theorem* plays an essential role in the sequel.

Theorem 2.2. *Let Ω be a domain in $\mathbb{R}^3$, let Γ_0 and Γ_1 be two relatively open subsets of Γ such that* $\mathrm{d}\Gamma$-meas $\Gamma_0 > 0$, $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, *and $\Gamma_0 \cap \Gamma_1 = \varnothing$, and let there be given a matrix field $\boldsymbol{e} \in \mathbb{L}^2(\Omega)$. Then there exists a vector field*

$$\boldsymbol{v} \in \boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega);\ \operatorname{tr}\boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_0\}$$

such that $\boldsymbol{e} = \nabla_s \boldsymbol{v}$ if and only if

$$\int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \quad \text{for all } \boldsymbol{\mu} \in \mathbb{M},$$

where the space $\mathbb{M}$ is defined as

$$\mathbb{M} := \{\boldsymbol{\mu} \in \mathbb{L}^2(\Omega);\ \mathbf{div}\,\boldsymbol{\mu} = \mathbf{0} \text{ in } \boldsymbol{H}^{-1}(\Omega),\ \langle \boldsymbol{\mu}\boldsymbol{\nu}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V}\}.$$

Besides, such a vector field $\boldsymbol{v} \in \boldsymbol{V}$ is uniquely defined.

3. Three different formulations of the displacement–traction problem of three-dimensional linearized elasticity as a minimization problem

Let Ω be a domain in $\mathbb{R}^3$ and let Γ_0 and Γ_1 be two relatively open subsets of $\Gamma := \partial\Omega$ that satisfy $\mathrm{d}\Gamma$-meas $\Gamma_0 > 0$, $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, and $\Gamma_0 \cap \Gamma_1 = \varnothing$. The following assumptions are made in the rest of the Note. The set $\overline{\Omega}$ is the reference configuration of a linearly elastic body, characterized by its *elasticity tensor field* $\mathbf{A} = (A_{ijk\ell})$ with components $A_{ijk\ell} \in L^\infty(\Omega)$ satisfying the symmetry relations $A_{ijk\ell} = A_{jik\ell} = A_{k\ell ij}$. The tensor field $\mathbf{A}$ is uniformly positive-definite almost-everywhere in Ω. Hence there exists a tensor field $\mathbf{B} = (B_{ijk\ell})$, called the *compliance tensor field*, that is the *inverse of* $\mathbf{A}$ and thus such that $B_{ijk\ell} \in L^\infty(\Omega)$ and $\mathbf{B}$ is also uniformly positive-definite almost-everywhere in Ω.

The body is subjected to *applied body forces* with density $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ in its interior and to *applied surface forces* of density $\boldsymbol{F} \in \boldsymbol{L}^2(\Gamma_1)$ on the portion Γ_1 of its boundary. The body is subjected to a *homogeneous boundary condition of place* along Γ_0.

The corresponding *displacement–traction problem*, or the *pure displacement problem* if $\Gamma_0 = \Gamma$, *of three-dimensional linearized elasticity* classically takes the form of the following minimization problem, where the minimizer $\bar{\boldsymbol{v}} : \overline{\Omega} \to \mathbb{R}^3$ is the *unknown displacement field*. That this minimization problem has one and only one solution is well known (see, e.g., Theorem 3.4 in Duvaut and Lions [6]).

Theorem 3.1 *(The classical displacement formulation). Let the space $\boldsymbol{V}$ be defined as in Theorem* 2.2. *Then there exists a unique vector field $\bar{\boldsymbol{v}} \in \boldsymbol{V}$ that satisfies*

$$J(\bar{\boldsymbol{v}}) = \inf_{\boldsymbol{v}\in\boldsymbol{V}} J(\boldsymbol{v}), \quad \text{where } J(\boldsymbol{v}) := \frac{1}{2}\int_\Omega \mathbf{A}\nabla_s\boldsymbol{v} : \nabla_s\boldsymbol{v}\,\mathrm{d}x - L(\boldsymbol{v}) \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V},$$

and

$$L(\boldsymbol{v}) := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{F}\cdot\boldsymbol{v}\,\mathrm{d}\Gamma \quad \text{for all } \boldsymbol{v} \in \boldsymbol{H}^1(\Omega).$$

It is then also classical (Brezzi and Fortin [3]) that the *stress tensor field* $\bar{\boldsymbol{\sigma}} := \mathbf{A}\nabla_s\bar{\boldsymbol{v}} \in \mathbb{L}^2(\Omega)$ inside the body can be also obtained as the solution of the following minimization problem.

Theorem 3.2 *(The classical stress formulation). Let the space $\boldsymbol{V}$ be defined as in Theorem 2.2. Then there exists a unique tensor field*

$$\bar{\boldsymbol{\sigma}} \in \mathbb{S} := \left\{\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}; \Omega);\ \mathbf{div}\,\boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} \text{ in } \boldsymbol{L}^2(\Omega),\ \langle \boldsymbol{\sigma}\boldsymbol{\nu} - \boldsymbol{F}, \operatorname{tr}\boldsymbol{v}\rangle_\Gamma = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{V}\right\},$$

that satisfies

$$g(\bar{\boldsymbol{\sigma}}) = \inf_{\sigma \in \mathbb{S}} g(\boldsymbol{\sigma}), \quad \text{where } g(\boldsymbol{\sigma}) := \frac{1}{2}\int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x \quad \text{for all } \boldsymbol{\sigma} \in \mathbb{L}^2(\Omega).$$

Besides, $\bar{\boldsymbol{\sigma}} = \mathbf{A}\nabla_s \bar{\boldsymbol{v}}$ in $\mathbb{L}^2(\Omega)$, where the vector field $\bar{\boldsymbol{v}} \in \boldsymbol{V}$ is the unique solution to the minimization problem of Theorem 3.1.

An *intrinsic approach* to the *same* displacement–traction problem consists in considering the *linearized strain tensor field* $\bar{\boldsymbol{e}} := \nabla_s \bar{\boldsymbol{v}} \in \mathbb{L}^2(\Omega)$ inside the body as the primary unknown, instead of the displacement field itself. One way to define such an approach is by means of Theorem 2.2, which leads to the following result (see Amrouche et al. [1] and Ciarlet et al. [4]).

Theorem 3.3 *(The strain formulation, a.k.a. the intrinsic approach). Let the space $\mathbb{M}$ be defined as in Theorem 2.2. Define the Hilbert space*

$$\mathbb{M}^\perp := \left\{\boldsymbol{e} \in \mathbb{L}^2(\Omega);\ \int_\Omega \boldsymbol{e} : \boldsymbol{\mu}\,\mathrm{d}x = 0 \text{ for all } \boldsymbol{\mu} \in \mathbb{M}\right\},$$

and, for each $\boldsymbol{e} \in \mathbb{M}^\perp$, let $\boldsymbol{\mathcal{F}}(\boldsymbol{e})$ denote the unique element in the space $\boldsymbol{V}$ that satisfies $\nabla_s \boldsymbol{\mathcal{F}}(\boldsymbol{e}) = \boldsymbol{e}$ (Theorem 2.2). Then the minimization problem: Find $\bar{\boldsymbol{e}} \in \mathbb{M}^\perp$ such that

$$j(\bar{\boldsymbol{e}}) = \inf_{\boldsymbol{e} \in \mathbb{M}^\perp} j(\boldsymbol{e}), \quad \text{where } j(\boldsymbol{e}) := \frac{1}{2}\int_\Omega \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x - L\big(\boldsymbol{\mathcal{F}}(\boldsymbol{e})\big),$$

has one and only one solution $\bar{\boldsymbol{e}}$. Besides, $\bar{\boldsymbol{e}} = \nabla_s \bar{\boldsymbol{v}}$, where the vector field $\bar{\boldsymbol{v}} \in \boldsymbol{V}$ is the unique solution to the minimization problem of Theorem 3.1.

4. The classical stress formulation of the displacement–traction problem as a point of departure

The minimization problem found in Theorem 3.2 constitutes our point of departure for constructing dual problems, by means of the approach described in Section 1. Accordingly, our first task consists in verifying that this formulation can be indeed recast in the abstract framework of Theorem 1.2.

Theorem 4.1. *Let the space $\boldsymbol{V}$ and the linear form $L \in \boldsymbol{V}^*$ be defined as in Theorem 3.1, let the mapping $\Lambda : \boldsymbol{\sigma} \in \mathbb{L}^2(\Omega) \to \Lambda\boldsymbol{\sigma} \in \boldsymbol{V}^*$ be defined by*

$${}_{\boldsymbol{V}^*}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\boldsymbol{V}} := \int_\Omega \boldsymbol{\sigma} : \nabla_s \boldsymbol{v}\,\mathrm{d}x \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V},$$

and finally, let the functions $g : \mathbb{L}^2(\Omega) \to \mathbb{R}$ and $h : \boldsymbol{V}^* \to \mathbb{R} \cup \{+\infty\}$ be respectively defined by

$$\boldsymbol{\sigma} \in \mathbb{L}^2(\Omega) \to g(\boldsymbol{\sigma}) := \frac{1}{2}\int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x,$$

$$\boldsymbol{v}^* \in \boldsymbol{V}^* \to h\big(\boldsymbol{v}^*\big) := 0 \quad \text{if } \boldsymbol{v}^* = L \quad \text{or} \quad h\big(\boldsymbol{v}^*\big) := +\infty \quad \text{if } \boldsymbol{v}^* \neq L.$$

Then $\Lambda \in \mathcal{L}(\mathbb{L}^2(\Omega); \boldsymbol{V}^*)$ and the functions g and h are both proper, convex, and lower semi-continuous. Define the function $G : \mathbb{L}^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by

$$G(\boldsymbol{\sigma}) := g(\boldsymbol{\sigma}) + h(\Lambda\boldsymbol{\sigma}) \quad \text{for all } \boldsymbol{\sigma} \in \mathbb{L}^2(\Omega).$$

Then

$$h(\Lambda\boldsymbol{\sigma}) = I_{\mathbb{S}}(\boldsymbol{\sigma}) \quad \text{for all } \boldsymbol{\sigma} \in \mathbb{L}^2(\Omega),$$

where the set $\mathbb{S}$ is defined as in Theorem 3.2, and the minimization problem of Theorem 3.2, viz., $\inf_{\boldsymbol{\sigma} \in \mathbb{S}} g(\boldsymbol{\sigma})$, is the same as the minimization problem

$$\inf_{\boldsymbol{\sigma} \in \mathbb{L}^2(\Omega)} G(\boldsymbol{\sigma}). \tag{$\mathcal{P}$}$$

Sketch of proof. It is easily seen that the mapping $\varLambda : \boldsymbol{\sigma} \in \mathbb{L}^2(\varOmega) \to \varLambda\boldsymbol{\sigma} \in \boldsymbol{V}^*$ is linear and continuous. The function $g : \mathbb{L}^2(\varOmega) \to \mathbb{R}$ is convex since the compliance tensor $\mathbf{B}$ is uniformly positive-definite almost-everywhere in $\varOmega$, and lower semi-continuous since g is continuous for the norm $\|\cdot\|_{\mathbb{L}^2(\varOmega)}$. The function $h : \boldsymbol{V}^* \to \mathbb{R} \cup \{+\infty\}$ is the indicator function of the subset $\{L\}$ of $\boldsymbol{V}^*$. Hence it is proper, convex because $\{L\}$ is a convex subset of $\boldsymbol{V}^*$, and lower semi-continuous because $\{L\}$ is a closed subset of $\boldsymbol{V}^*$. The rest of the proof makes an essential use of the Green formula of Theorem 2.1; for details, see [5]. □

In view of identifying the dual problems of the minimization problem $(\mathcal{P})$ of Theorem 4.1, it remains to identify the Legendre–Fenchel transforms (Section 1) of the functions h and g introduced in this theorem. The next result is known (see, e.g., [7]).

Theorem 4.2. *The Legendre–Fenchel transforms $g^* : \mathbb{L}^2(\varOmega) \to \mathbb{R}$ and $h^* : \boldsymbol{V} \to \mathbb{R}$ of the functions $g : \mathbb{L}^2(\varOmega) \to \mathbb{R}$ and $h : \boldsymbol{V}^* \to \mathbb{R} \cup \{+\infty\}$ defined in Theorem* 4.1 *are respectively given by*

$$g^*(\boldsymbol{e}) := \frac{1}{2}\int_{\varOmega} \mathbf{A}\boldsymbol{e} : \boldsymbol{e}\,\mathrm{d}x \quad \textit{for all } \boldsymbol{e} \in \mathbb{L}^2(\varOmega) \quad \textit{and} \quad h^*(\boldsymbol{v}) := \int_{\varOmega} \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\varOmega} \boldsymbol{F}\cdot\boldsymbol{v}\,\mathrm{d}\varGamma = L(\boldsymbol{v}) \quad \textit{for all } \boldsymbol{v} \in \boldsymbol{V}.$$

5. A first dual problem to the stress formulation

Following the approach described in Section 1, we now *identify the first dual formulation* $(\mathcal{P}^*)$ to the stress formulation of the displacement–traction problem, formulated for this purpose in the form of the equivalent minimization problem $(\mathcal{P})$ described in Theorem 4.1; for the proof, see [5].

Theorem 5.1. *Consider the minimization problem $(\mathcal{P})$ of Theorem* 4.1, *viz.*, $\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\varOmega)} G(\boldsymbol{\sigma})$. *Let*

$$G^*(\boldsymbol{e}) := \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\varOmega)} \left\{ \int_{\varOmega} \boldsymbol{e} : \boldsymbol{\sigma}\,\mathrm{d}x + h(\varLambda\boldsymbol{\sigma}) \right\} - g^*(\boldsymbol{e}) \quad \textit{for each } \boldsymbol{e} \in \mathbb{L}^2(\varOmega),$$

where $g^* : \mathbb{L}^2(\varOmega) \to \mathbb{R}$ is the Legendre–Fenchel transform of the function g, and let

$$\sup_{\boldsymbol{e}\in\mathbb{L}^2(\varOmega)} G^*(\boldsymbol{e}), \tag{$\mathcal{P}^*$}$$

be the corresponding dual problem. Let the space $\mathbb{M}^\perp$ and the functional $j : \mathbb{M}^\perp \to \mathbb{R}$ be defined as in Theorem 3.3. *Then the dual problem $(\mathcal{P}^*)$ can be also written as*

$$\sup_{\boldsymbol{e}\in\mathbb{L}^2(\varOmega)} G^*(\boldsymbol{e}) = -\inf_{\boldsymbol{e}\in\mathbb{M}^\perp} j(\boldsymbol{e}).$$

Besides,

$$G(\bar{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\varOmega)} G(\boldsymbol{\sigma}) = \sup_{\boldsymbol{e}\in\mathbb{L}^2(\varOmega)} G^*(\boldsymbol{e}) = G^*(\bar{\boldsymbol{e}}),$$

where $\bar{\boldsymbol{\sigma}} \in \mathbb{S} \subset \mathbb{L}^2(\varOmega)$ and $\bar{\boldsymbol{e}} \in \mathbb{M}^\perp \subset \mathbb{L}^2(\varOmega)$ are the solutions of the minimization problems of Theorems 3.2 *and* 3.3.

Define the Lagrangian $\mathcal{L} : \mathbb{L}^2(\varOmega) \times \mathbb{L}^2(\varOmega) \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{L}(\boldsymbol{\sigma}, \boldsymbol{e}) := \int_{\varOmega} \boldsymbol{e} : \boldsymbol{\sigma}\,\mathrm{d}x - g^*(\boldsymbol{e}) + h(\varLambda\boldsymbol{\sigma}) \quad \textit{for all } (\boldsymbol{\sigma}, \boldsymbol{e}) \in \mathbb{L}^2(\varOmega) \times \mathbb{L}^2(\varOmega).$$

Then

$$\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\varOmega)} \sup_{\boldsymbol{e}\in\mathbb{L}^2(\varOmega)} \mathcal{L}(\boldsymbol{\sigma}, \boldsymbol{e}) = \mathcal{L}(\bar{\boldsymbol{\sigma}}, \bar{\boldsymbol{e}}) = \sup_{\boldsymbol{e}\in\mathbb{L}^2(\varOmega)} \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\varOmega)} \mathcal{L}(\boldsymbol{\sigma}, \boldsymbol{e}).$$

Theorem 5.1 thus shows that *the dual problem $(\mathcal{P}^*)$ of the stress formulation of the displacement–traction problem of linearized elasticity* (Theorem 3.2) *is, up to a change of sign, the strain formulation of the same problem* (Theorem 3.3).

6. A second dual problem to the stress formulation

We now *identify the second dual formulation* $(\widetilde{\mathcal{P}}^*)$ to the stress formulation of the displacement problem, again formulated as the problem $(\mathcal{P})$ of Theorem 4.1; for the proof, see [5].

Theorem 6.1. *Consider the minimization problem* $(\mathcal{P})$ *of Theorem* 4.1, *viz.*, $\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\Omega)} G(\boldsymbol{\sigma})$. *Let*

$$\widetilde{G}^*(\boldsymbol{v}) := \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\Omega)} \left\{ \frac{1}{2}\int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x + {}_{\boldsymbol{V}^*}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\boldsymbol{V}} \right\} - h^*(\boldsymbol{v}) \quad \text{for each } \boldsymbol{v}\in\boldsymbol{V},$$

where $h^* : \boldsymbol{V} \to \mathbb{R}$ *is the Legendre–Fenchel transform of the function h, and let*

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{G}^*(\boldsymbol{v}) \tag{$\widetilde{\mathcal{P}}^*$}$$

be the corresponding dual problem. Let the functional $J : \boldsymbol{V} \to \mathbb{R}$ *be defined as in Theorem* 3.1. *Then the dual problem* $(\widetilde{\mathcal{P}}^*)$ *can be also written as*

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{G}^*(\boldsymbol{v}) = -\inf_{\boldsymbol{v}\in\boldsymbol{V}} J(\boldsymbol{v}).$$

Besides,

$$G(\overline{\boldsymbol{\sigma}}) = \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\Omega)} G(\boldsymbol{\sigma}) = \sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{G}^*(\boldsymbol{v}) = G^*(-\overline{\boldsymbol{v}}),$$

where $\overline{\boldsymbol{\sigma}} \in \mathbb{S} \subset \mathbb{L}^2(\Omega)$ *and* $\overline{\boldsymbol{v}} \in \boldsymbol{V}$ *are the solutions of the minimization problems of Theorems* 3.2 *and* 3.1.

Define the Lagrangian $\widetilde{\mathcal{L}} : \mathbb{L}^2(\Omega) \times \boldsymbol{V} \to \mathbb{R}$ *by*

$$\widetilde{\mathcal{L}}(\boldsymbol{\sigma}, \boldsymbol{v}) := \frac{1}{2}\int_\Omega \mathbf{B}\boldsymbol{\sigma} : \boldsymbol{\sigma}\,\mathrm{d}x + {}_{\boldsymbol{V}^*}\langle \Lambda\boldsymbol{\sigma}, \boldsymbol{v}\rangle_{\boldsymbol{V}} - h^*(\boldsymbol{v}) \quad \text{for all } (\boldsymbol{\sigma}, \boldsymbol{v}) \in \mathbb{L}^2(\Omega)\times\boldsymbol{V}.$$

Then

$$\inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\Omega)} \sup_{\boldsymbol{v}\in\boldsymbol{V}} \widetilde{\mathcal{L}}(\boldsymbol{\sigma}, \boldsymbol{v}) = \widetilde{\mathcal{L}}(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{v}}) = \sup_{\boldsymbol{v}\in\boldsymbol{V}} \inf_{\boldsymbol{\sigma}\in\mathbb{L}^2(\Omega)} \widetilde{\mathcal{L}}(\boldsymbol{\sigma}, \boldsymbol{v}).$$

Theorem 6.1 thus shows that *the dual problem* $(\widetilde{\mathcal{P}}^*)$ *to the stress formulation of the displacement–traction problem of linearized elasticity* (Theorem 3.2) *is, up to a change of sign, the displacement formulation of the same problem* (Theorem 3.1).

7. Concluding remarks

The strain formulation of, a.k.a. the intrinsic approach to, the displacement–traction problem described in Theorem 3.3 was derived *a priori*, in [1] or [4], as a way to re-formulate this problem as a quadratic minimization problem with the strain tensor field as the sole unknown. One main conclusion to be drawn from the present analysis is thus that *this strain formulation may be also viewed as a Legendre–Fenchel dual problem to the classical stress formulation* (Theorem 5.1). This constitutes the main novelty of this Note.

Another novelty is that *the classical displacement formulation can be also viewed as a Legendre–Fenchel dual problem to the same classical stress formulation* (Theorem 6.1).

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