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Partial Differential Equations/Mathematical Problems in Mechanics

Existence of global weak solutions for a viscous 2D bilayer Shallow Water model

Existence de solutions faibles globales d'un modèle bicouche bidimensionnel de Saint-Venant

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A R T I C L E I N F O

A B S T R A C T

We consider a system composed by two immiscible fluids in two-dimensional space that can be modelized by a bilayer Shallow Water equations with extra friction terms and capillary effects. We give an existence theorem of global weak solutions in a periodic domain.

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R É S U M É

Nous considérons un système composé par deux fluides immiscibles dans un domaine bidimensionnel pouvant être représenté par un modèle bicouche visqueux de Saint-Venant avec des termes de friction additionnels et des effets de capillarité. Nous donnons un théorème d'existence de solutions faibles globales dans un domaine périodique.

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Version française abrégée

Dans cette Note, nous nous intéressons à l'étude de l'existence de solutions faibles globales en temps d'un modèle bicouche visqueux de Saint-Venant dérivé dans [6]. Notons que dans le cas d'une couche, dans [1] et [3] les auteurs ont obtenu l'existence de solutions faibles grâce à une nouvelle entropie (BD) introduite premièrement par Bresch et Desjardins dans [1]. On peut trouver d'autres résultats sur l'existence de solutions pour des modèles bicouche de Saint-Venant dans [4] et [5]. Dans ces modèles, les termes couplant les deux fluides compliquent le passage à la limite. Dans [7] une étude du modèle bicouche mais où les termes de friction ont été simplifiés a été faite. Les termes de friction couplant les deux couches dans le présent cas sont plus difficiles à contrôler et compliquent également le passage à la limite dans les suites de solutions approchées. Notons que les termes de viscosité ici sont de la forme $\mathrm{div}(h_i D(u_i))$ et permettent comme dans [1,3] et [6] d'avoir une plus grande régularité sur les hauteurs.

Plus précisément, nous considérons dans un domaine périodique bidimensionnel le modèle (1)–(4) ci-dessous. Pour ce modèle, nous obtenons le théorème suivant :

Théorème 0.1. *Il existe une solution faible globale de* (1)–(4) *satisfaisant les inégalités d'énergies.*

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1. Introduction

In this Note, we are interested in the study of the existence of global weak solution for a 2D bilayer Shallow-Water model derived in [6]. In the one layer case with degenerate viscous term the global existence of weak solutions is obtained in [1] and [3] thanks to a new entropy inequality namely BD entropy derived first by Bresch and Desjardins in [1]. Notice that other bilayer Shallow Water models are studied in [4] and [5]. The main difficulty in these papers arises from the terms coupling the two layers. We can also find in [7] the theoretical study of the bilayer model but where the friction terms have been simplified. The friction terms considered here are more difficult to control and to pass to the limit in the weak formulation. Such work will be the aim of this paper.

After these lines we introduce the model to study and we state how a weak solution is defined for our system. In Section 2 we show the energies inequalities that allow us to prove the stability of weak solution, for their construction one can follow the lines of [2]. The existence result is presented in the last section, where we also give the schedule of the proof.

We consider in a two-dimensional periodic domain Ω, a system composed by two layers of immiscible fluids with different and constant densities ρ_i and viscosities μ_i, for $i = 1, 2$. The unknowns are the velocities v_i and the thickness h_i, respectively. The index 1 design the deeper layer and the index 2, the upper layer of the flow. We assume that the friction coefficient at the bottom α and the tension coefficients α_1, α_2 (on the interface and on the free surface, respectively) are positive. The model reads as follows:

$$\partial_t h_1 + \operatorname{div}(h_1 v_1) = 0; \tag{1}$$

$$\rho_1 \partial_t(h_1 v_1) + \rho_1 \operatorname{div}(h_1 v_1 \otimes v_1) - 2\mu_1 \operatorname{div}(h_1 D(v_1))$$
$$+ \rho_1 g h_1 \nabla h_1 + \rho_2 g h_1 \nabla h_2 + \delta^{-1}\gamma\left(\beta\frac{\alpha h_1}{6\mu_1}v_1 + (v_1 - v_2)\right)$$
$$+ \beta\alpha\left(v_1 + \delta^{-1}\frac{\gamma h_1}{6\mu_1}(v_1 - v_2)\right) - \alpha_1 h_1 \nabla(\Delta h_1) - \alpha_2 h_1 \nabla(\Delta h_2) = 0; \tag{2}$$

$$\partial_t h_2 + \operatorname{div}(h_2 v_2) = 0; \tag{3}$$

$$\rho_2 \partial_t(h_2 v_2) + \rho_2 \operatorname{div}(h_2 v_2 \otimes v_2) - 2\mu_2 \operatorname{div}(h_2 D(v_2))$$
$$+ \rho_2 g h_2 \nabla h_2 + \rho_2 g h_2 \nabla h_1 - \delta^{-1}\gamma\left(\beta\frac{\alpha h_1}{6\mu_1}v_1 + (v_1 - v_2)\right) - \alpha_2 h_2 \nabla(\Delta(h_1 + h_2)) = 0 \tag{4}$$

with initial conditions: $h_{i|t=0} = h_{i_0} \geqslant 0$, $h_i v_{i|t=0} = m_{i_0}$, for which we assume the following regularity: $h_{i_0} \in L^2(\Omega)$, $\nabla h_{i_0} \in (L^2(\Omega))^2$, $\nabla\sqrt{h_{i_0}} \in (L^2(\Omega))^2$, $\frac{|m_{i_0}|^2}{h_{i_0}} \in L^1(\Omega)$, $\log_-(h_{i_0}) \in L^1(\Omega)$. The functions $\beta = \beta(h_1)$, $\delta = \delta(h_1, h_2)$ and $\gamma = \gamma(h_1, h_2)$ are the drag coefficients given by:

$$\beta(h_1) = \left(1 + \frac{\alpha}{3\mu_1}h_1\right)^{-1}, \qquad \delta = \frac{3\mu_1\mu_2 + \gamma\mu_2 h_1 + \gamma\mu_1 h_2}{3\mu_1\mu_2} \quad \text{and} \quad \gamma = \frac{h_1 h_2}{\frac{\rho_1}{\rho_2}h_1 + \frac{\rho_2}{\rho_1}h_2}.$$

The terms $D(v) = \frac{\nabla v + \nabla^t v}{2}$ and $A(v) = \frac{\nabla v - \nabla^t v}{2}$ are respectively the deformation and the vorticity tensors.

Definition 1.1. We shall say that (h_1, h_2, v_1, v_2) is a weak solution of (1)–(4) if (1) and (3) hold in $(\mathcal{D}'(0, T) \times \Omega)^2$; $h_{1|t=0} = h_1^0 \geqslant 0$ and $h_{2|t=0} = h_2^0 \geqslant 0$ in $\mathcal{D}'(\Omega)$, the following assumptions are satisfied:

$$h_i \in L^\infty(0, T; L^2(\Omega)); \qquad \nabla h_i, \quad \sqrt{h_i}v_i \in L^\infty(0, T; (L^2(\Omega))^2); \qquad v_1 \in L^2(0, T; (L^2(\Omega))^2);$$
$$\sqrt{h_i}D(v_i) \in L^2(0, T; (L^2(\Omega))^4); \qquad \sqrt{B(h_1, h_2)}(v_1 - v_2) \in L^2(0, T; (L^2(\Omega))^2);$$
$$\nabla\sqrt{h_{i_n}} \in L^2(0, T; L^2(\Omega)^2); \qquad \Delta h_{i_n} \in L^2(0, T; L^2(\Omega)); \tag{5}$$

and for any $\varphi \in \mathcal{C}^\infty((0, T) \times \Omega)^2$ with $\varphi(T, \cdot) = 0$ (φ with compact support), we have:

$$-\int_\Omega \rho_1 h_1^0 v_1^0 h_1^0 \varphi(0, \cdot) - \int_0^T\int_\Omega \rho_1 h_1^2 v_1 \partial_t\varphi + \rho_1\int_0^T\int_\Omega h_1^2(v_1 \cdot \varphi)\operatorname{div}v_1 - \rho_1\int_0^T\int_\Omega (h_1 v_1 \otimes h_1 v_1) : D(\varphi)$$

$$+ 2\mu_1\int_0^T\int_\Omega h_1^2(D(v_1) : D(\varphi)) + 2\mu_1\int_0^T\int_\Omega h_1(D(v_1) : (\nabla h_1 \otimes \varphi)) + \int_0^T\int_\Omega \delta^{-1}\gamma\left(\beta\frac{\alpha h_1}{6\mu_1}v_1 + (v_1 - v_2)\right)h_1\varphi$$

$$+ \int\limits_0^T\!\!\int\limits_\Omega \beta\alpha\left(v_1 + \delta^{-1}\gamma\frac{h_1}{6\mu_1}(v_1 - v_2)\right)h_1\varphi + \frac{1}{2}\rho_1 g\int\limits_0^T\!\!\int\limits_\Omega h_1^3\,\mathrm{div}\,\varphi$$

$$+ \frac{1}{2}\rho_1 g\int\limits_0^T\!\!\int\limits_\Omega h_1^2(\varphi\cdot\nabla h_1) + \rho_2 g\int\limits_0^T\!\!\int\limits_\Omega h_1^2(\varphi\cdot\nabla h_2) + \alpha_1\int\limits_0^T\!\!\int\limits_\Omega h_1^2\Delta h_1\,\mathrm{div}\,\varphi$$

$$+ 2\alpha_1\int\limits_0^T\!\!\int\limits_\Omega h_1\Delta h_1(\varphi\cdot\nabla h_1) + \alpha_2\int\limits_0^T\!\!\int\limits_\Omega h_1^2\Delta h_2\,\mathrm{div}\,\varphi + \alpha_2\int\limits_0^T\!\!\int\limits_\Omega h_1(\varphi\cdot\nabla h_1)\Delta h_2 = 0 \tag{6}$$

and

$$-\int\limits_\Omega \rho_2 h_2^0 v_2^0 h_2^0\varphi(0,\cdot) - \int\limits_0^T\!\!\int\limits_\Omega \rho_2 h_2^2 v_2\partial_t\varphi + \rho_2\int\limits_0^T\!\!\int\limits_\Omega h_2^2(v_2\cdot\varphi)\,\mathrm{div}\,v_2 - \rho_2\int\limits_0^T\!\!\int\limits_\Omega (h_2 v_2\otimes h_2 v_2):D(\varphi)$$

$$+ 2\mu_2\int\limits_0^T\!\!\int\limits_\Omega h_2^2\big(D(v_2):D(\varphi)\big) + 2\mu_2\int\limits_0^T\!\!\int\limits_\Omega h_2\big(D(v_2):(\nabla h_2\otimes\varphi)\big) - \int\limits_0^T\!\!\int\limits_\Omega \delta^{-1}\gamma\left(\beta\frac{\alpha h_1}{6\mu_1}v_1 + (v_1 - v_2)\right)h_2\varphi$$

$$+ \frac{1}{2}\rho_2 g\int\limits_0^T\!\!\int\limits_\Omega h_2^3\,\mathrm{div}\,\varphi + \frac{1}{2}\rho_2 g\int\limits_0^T\!\!\int\limits_\Omega h_2^2(\varphi\cdot\nabla h_2) + \rho_2 g\int\limits_0^T\!\!\int\limits_\Omega h_2^2(\varphi\cdot\nabla h_1)$$

$$+ \alpha_2\int\limits_0^T\!\!\int\limits_\Omega h_2^2\Delta(h_1 + h_2)\,\mathrm{div}\,\varphi + 2\alpha_2\int\limits_0^T\!\!\int\limits_\Omega h_2\Delta(h_1 + h_2)(\varphi\cdot\nabla h_2) = 0. \tag{7}$$

2. Energy estimates

Proposition 2.1. *Let* (h_1, h_2, v_1, v_2) *be a solution of the system* (1)–(4)*. Then, the following inequality holds*:

$$\frac{1}{2}\rho_1\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega h_1|v_1|^2 + \frac{1}{2}\rho_2\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega h_2|v_2|^2 + 2\mu_1\int\limits_\Omega h_1\big(D(v_1):D(v_1)\big) + 2\mu_2\int\limits_\Omega h_2\big(D(v_2):D(v_2)\big)$$

$$+ \frac{1}{2}g(\rho_1 - \rho_2)\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega |h_1|^2 + \int\limits_\Omega\left(\delta^{-1}\gamma\beta\frac{\alpha h_1}{3\mu_1} + \alpha\beta\right)|v_1|^2 + \frac{1}{2}\rho_2 g\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega |h_1 + h_2|^2$$

$$+ \frac{1}{2}(\alpha_1 - \alpha_2)\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega |\nabla h_1|^2 + \frac{\alpha_2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \big|\nabla(h_1 + h_2)\big|^2 + \int\limits_\Omega \delta^{-1}\gamma|v_1 - v_2|^2$$

$$\leqslant \frac{1}{2}\frac{\rho_1}{\rho_2}\int\limits_\Omega h_1|v_1|^2 + \frac{1}{2}\frac{\rho_2}{\rho_1}\int\limits_\Omega h_2|v_2|^2. \tag{8}$$

Proposition 2.2. *If we assume that* (h_1, h_2, v_1, v_2) *is a smooth solution of system* (1)–(4)*, then*

$$\frac{1}{2}\rho_2\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega h_1|\rho_1 v_1 + 2\mu_1\nabla\log h_1|^2 + \frac{1}{2}\rho_1\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega h_2|\rho_2 v_2 + 2\mu_2\nabla\log h_2|^2$$

$$+ \rho_1\rho_2\left(\frac{1}{2}g(\rho_1 - \rho_2)\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega |h_1|^2 + \frac{1}{2}\rho_2 g\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega |h_1 + h_2|^2 + \left(\delta^{-1}\gamma\beta\frac{\alpha h_1}{3\mu_1} + \alpha\beta\right)|v_1|^2\right)$$

$$+ 2\mu_2\rho_1\rho_2\int\limits_\Omega h_2\big(A(v_2):A(v_2)\big) + 2\mu_1\rho_1\rho_2\int\limits_\Omega h_1\big(A(v_1):A(v_1)\big) + \rho_1\rho_2\int\limits_\Omega \delta^{-1}\gamma|v_1 - v_2|^2$$

$$+ \frac{1}{2}\rho_1\rho_2(\alpha_1 - \alpha_2)\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega |\nabla h_1|^2 + \frac{1}{2}\alpha_2\rho_1\rho_2\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \big|\nabla(h_1 + h_2)\big|^2 + 2\mu_1\rho_1\rho_2 g\int\limits_\Omega |\nabla h_1|^2$$

$$+ 2\mu_2\rho_1\rho_2 g \int_\Omega |\nabla h_2|^2 + 2\mu_1\alpha_1\rho_2 \int_\Omega |\Delta h_1|^2 + 2\mu_2\alpha_2\rho_1 \int_\Omega |\Delta h_2|^2 - 2\mu_1\alpha\rho_2 \frac{d}{dt}\int_\Omega \log\left(\frac{h_1}{3\mu_1 + \alpha h_1}\right)$$

$$+ 2\mu_1\alpha\rho_2 \int_\Omega \beta'(h_1)v_1\nabla h_1 + 2\rho_2 g(\rho_2\mu_1 + \rho_1\mu_2) \int_\Omega \nabla h_1\nabla h_2 + 2\alpha_2(\rho_2\mu_1 + \rho_1\mu_2) \int_\Omega \Delta h_1\Delta h_2$$

$$+ 2\mu_1\rho_2 \int_\Omega \delta^{-1}\gamma\left(1 + \beta\frac{\alpha h_1}{6\mu_1}\right)(v_1 - v_2)\cdot\nabla\log h_1 + 2\mu_1\rho_2 \int_\Omega \delta^{-1}\gamma\beta\frac{\alpha h_1}{6\mu_1}v_1\cdot\nabla\log h_1$$

$$- 2\mu_2\rho_1 \int_\Omega \delta^{-1}\gamma\beta\frac{\alpha h_1}{6\mu_1}v_1\cdot\nabla\log h_2 - 2\mu_2\rho_1 \int_\Omega \delta^{-1}\gamma(v_1 - v_2)\cdot\nabla\log h_2$$

$$\leqslant \frac{1}{6}\rho_1\left(\frac{3}{\rho_2} + \alpha\right)\int_\Omega h_1|v_1|^2 + \frac{1}{2}\rho_2^2\int_\Omega h_2|v_2|^2 + \frac{1}{3}\rho_1\alpha\int_\Omega h_1|\nabla h_1|^2. \tag{9}$$

Remark 1. The main difficulty in the proof of the two preceding inequalities is located in the terms coming from the friction effects between the two fluids and between the lower fluid and the bottom. They are those we are going to focus on. The rest of the terms are controlled as in [7]. Note that:

- $F_1 = 2\mu_1\rho_2\int_\Omega \delta^{-1}\gamma\beta\frac{\alpha h_1}{6\mu_1}v_1\cdot\nabla\log h_1$. And we use the fact that $\delta^{-1} \leqslant 1$, $\beta \leqslant 1$ and $\gamma \leqslant \frac{\rho_1}{\rho_2}h_1$ to get:

$$F_1 \leqslant \frac{\rho_1\alpha}{6}\left(\int_\Omega h_1|v_1|^2 + \int_\Omega h_1|\nabla h_1|^2\right).$$

 Next, since $\frac{\alpha\beta}{6\mu_1}h_1 \leqslant \frac{1}{2}$, we have

- $$F_2 = 2\mu_1\rho_2\int_\Omega \delta^{-1}\gamma\left(1 + \beta\frac{\alpha h_1}{6\mu_1}\right)(v_1 - v_2)\cdot\nabla\log h_1 \leqslant \frac{3}{2}\mu_1\rho_2\int_\Omega \delta^{-1}\gamma|v_1 - v_2|^2 + 6\mu_1\rho_1\int_\Omega |\nabla\sqrt{h_1}|^2.$$

 And similarly the two last terms:

- $$F_3 = 2\mu_2\rho_1\int_\Omega \delta^{-1}\gamma\beta\frac{\alpha h_1}{6\mu_1}v_1\cdot\nabla\log h_2 \leqslant \mu_2\rho_1\int_\Omega \delta^{-1}\gamma\beta\frac{\alpha h_1}{6\mu_1}|v_1|^2 + 2\mu_2\rho_2\int_\Omega |\nabla\sqrt{h_2}|^2;$$

- $$F_4 = 2\mu_2\rho_1\int_\Omega \delta^{-1}\gamma(v_1 - v_2)\cdot\nabla\log h_2 \leqslant \frac{1}{2}\mu_2\rho_1\int_\Omega \delta^{-1}\gamma|v_1 - v_2|^2 + 2\mu_2\rho_2\int_\Omega |\nabla\sqrt{h_2}|^2.$$

3. Main result

Theorem 3.1. *There exists a global weak solution* (h_1, h_2, v_1, v_2) *of* (1)–(4) *satisfying the energy inequalities* (8) *and* (9).

Proof. We consider a sequence of approximate weak solutions namely $(h_{1_n}, h_{2_n}, v_{1_n}, v_{2_n})$ satisfying the two entropy inequalities. Such approximate solutions can be constructed with enough regularity along the lines of [2]. We assume that its initial data satisfy (C independent of n): $h_{i_n}^0 \to h_i^0$ in $L^1(\Omega)$, $h_{i_n}^0 v_{i_n}^0 \to h_i^0 v_i^0$ in $L^1(\Omega)$, $\int_\Omega h_{i_n}^0|v_{i_n}^0|^2 + |h_{i_n}^0|^2 < C$, $\int_\Omega \nabla(h_{i_n}^0)^{1/2} < C$ $i = 0, 1$. The proof of Theorem 3.1 is the same as that built in [7]. We must only prove some additional strong convergences. We give first the preliminary convergence.

Preliminary convergence. For $i = 1, 2$, we have h_{i_n} bounded in $L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$. Moreover $\partial_t h_{i_n} = -\operatorname{div}(h_{i_n}v_{i_n})$ is bounded in $L^2(0, T; H^{-1}(\Omega))$. Thanks to the Sobolev's imbedding, we have $\forall s \in (0, 1)$, $H^1(\Omega) \hookrightarrow H^s(\Omega)$, $H^s(\Omega) \subset H^{-1}(\Omega)$, $H^2(\Omega) \hookrightarrow H^{1+s}(\Omega)$, $H^{1+s}(\Omega) \subset H^{-1}(\Omega)$. Then, up to the extraction of a subsequence, there exists $h_i \in L^2(0, T; H^2(\Omega))$ ($i = 1, 2$) such that ∇h_i and $\nabla\sqrt{h_i}$ belong to $L^\infty(0, T; L^2(\Omega)^2)$ and $h_{i_n} \longrightarrow h_i$ in $L^p(0, T; H^{1+1/p}(\Omega)) \cap \mathcal{C}([0, T]; H^{1/p}(\Omega))$, $\forall p \in (2, +\infty)$. Next, since $\sqrt{h_{i_n}}v_{i_n}$ is bounded in $L^\infty(0, T; (L^2(\Omega))^2)$, we deduce that it converges weakly in $L^2(0, T; (L^2(\Omega))^2)$ up to a subsequence to some limit $z_i \in L^2(0, T; (L^2(\Omega))^2)$.

Let us define v_i to be:

$$v_i = \begin{cases} \frac{z_i}{\sqrt{h_i}} & \text{if } h_i > 0; \\ 0 & \text{if } h_i = 0. \end{cases} \tag{10}$$

To prove the convergence of $(h_{i_n}v_{i_n})$, we write it as $h_{i_n}v_{i_n} = \sqrt{h_{i_n}}v_{i_n}\sqrt{h_{i_n}}$. Notice that $\sqrt{h_{i_n}}v_{i_n} \rightharpoonup z_i$ in $L^2(0, T; (L^2(\Omega))^2)$, and so, we prove now the strong convergence for $(\sqrt{h_{i_n}})$ in $L^p(0, T; L^p(\Omega))$. This proof is given as follows: we have

$\|\nabla\sqrt{h_{i_n}}\|_{L^2(0,T;(L^2(\Omega))^2)} \leqslant C$ and $\|\sqrt{h_{i_n}}\|_{L^\infty(0,T;(L^4(\Omega))^2)} \leqslant C$, so, we can write that $(\sqrt{h_{i_n}})$ is bounded in $L^\infty(0,T;H^1(\Omega))$ and $\partial_t\sqrt{h_{i_n}} = \frac{1}{2}\sqrt{h_{i_n}}\operatorname{div} v_{i_n} - \frac{1}{2}\operatorname{div}(\sqrt{h_{i_n}}v_{i_n})$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. Consequently, the subsequence, $(\sqrt{h_{i_n}})$ converges strongly to some $\sqrt{h_i}$ in $L^p(0,T;L^p(\Omega))$. We then deduce that

$$h_{i_n}v_{i_n} \rightharpoonup \sqrt{h_i}z_i = h_i v_i \quad \text{in } L^2(0,T;L^2(\Omega)^2). \tag{11}$$

So we can pass to the limit in the mass equations. We give now some strong convergences which are necessary to pass to the limit in the momentum equations, more precisely in the frictions terms. For the others terms, see [7].

Additional convergence. Note that $|\delta_n^{-1} - \delta^{-1}| \leqslant \mu_2(|\gamma_n||h_{1_n} - h_1| + |h_1||\gamma_n - \gamma|) + \mu_1(|\gamma_n||h_{2_n} - h_2| + |h_2||\gamma_n - \gamma|)$. These terms have all them the same convergence behavior. So, by Hölder inequality, we have

$$\int_\Omega |\gamma_n|^{p/2}|h_{i_n} - h_i|^{p/2} \leqslant \int_\Omega |\gamma_n|^p \int_\Omega |h_{i_n} - h_i|^p \quad \text{and} \quad \int_\Omega |h_i|^{p/2}|\gamma_n - \gamma|^{p/2} \leqslant \int_\Omega |h_i|^p \int_\Omega |\gamma_n - \gamma|^p.$$

Since (h_{i_n}) and (γ_n) are bounded in $L^p(0,T;L^p(\Omega))$, we find the strong convergence of (δ_n^{-1}) to δ^{-1} in $L^{p/2}(0,T;L^{p/2}(\Omega))$. Next we have $\|\frac{\gamma_n}{\sqrt{h_{2_n}}} - \frac{\gamma}{\sqrt{h_2}}\|_{L^3(0,T;L^3(\Omega))}^3 \leqslant \int_0^T \int_\Omega (a_1 + a_2 + a_3 + a_4)^3 \, dx$, being $a_1 = \frac{\rho_2}{\rho_1}|\sqrt{h_{2_n}} - \sqrt{h_2}|$, $a_2 = \frac{\rho_2}{\rho_1}h_{1_n}h_2|\sqrt{h_{2_n}} - \sqrt{h_2}|$, $a_3 = \frac{\rho_2}{\rho_1}h_2^{3/2}|h_{1_n} - h_1|$, $a_4 = \frac{\rho_2}{\rho_1}h_1\sqrt{h_2}|h_2 - h_{2_n}|$.

There exist some constant $M_q \geqslant 0$ ($1 \leqslant q \leqslant 8$) such that $|\sum_{i=1}^4 a_i|^3 \leqslant \sum_{i=1}^4 \sum_{q=1}^8 M_q a_i^q$. It's enough to prove the strong convergence of $\int_0^T \int_\Omega |a_i|^q$ for all $1 \leqslant i \leqslant 4$ and $1 \leqslant q \leqslant 8$. More precisely, the difficulties appear only for $i = 2, 4$. We give the proof of the convergence of a_2. The rest follows the same way. To do this, we write:

$$\int_0^T \int_\Omega |h_{1_n}h_2|^q |\sqrt{h_{2_n}} - \sqrt{h_2}|^q \leqslant \int_0^T \left(\int_\Omega |h_{1_n}h_2|^{2q}\right)^{1/2} \int_0^T \left(\int_\Omega |\sqrt{h_{2_n}} - \sqrt{h_2}|^{2q}\right)^{1/2}$$

for all $q > 1$. Hence, $\int_0^T \int_\Omega |h_{1_n}h_2|^q |\sqrt{h_{2_n}} - \sqrt{h_2}|^q \leqslant \|h_{1_n}h_2\|_{L^{2q}(0,T;L^{2q}(\Omega))} \|\sqrt{h_{2_n}} - \sqrt{h_2}\|_{L^{2q}(0,T;L^{2q}(\Omega))}$. But $\|h_{1_n}h_2\|_{L^{2q}(0,T;L^{2q}(\Omega))}$ is bounded and $(\sqrt{h_{2_n}})$ converges strongly to $(\sqrt{h_2})$ in $L^{2q}(0,T;L^{2q}(\Omega))$. Finally, we get the strong convergence of $(\frac{\gamma_n}{\sqrt{h_{2_n}}})$ to $\frac{\gamma}{\sqrt{h_2}}$ in $L^3(0,T;L^3(\Omega))^3$.

These convergences allow us to pass to the limit in the friction terms.

Acknowledgements

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