Partial Differential Equations/Numerical Analysis

Variational forms for the inverses of integral logarithmic operators over an interval

Formulations variationnelles pour les inverses des opérateurs intégraux logarithmiques définis sur un intervalle

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\textbf{Abstract}

We present explicit and exact variational formulations for the weakly singular and hypersingular operators over an interval as well as for their corresponding inverses. By decomposing the solutions in symmetric and antisymmetric parts, we precisely characterize the associated Sobolev spaces. Moreover, we are able to define novel Calderón-type identities in each case.

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We consider the Laplace problem with two different Dirichlet conditions \( g^\pm \) from above and below on \( \Gamma_c \). This boundary data lies in the Hilbert space:

\[
\mathcal{X} := \{ g = (g^+, g^-) \in H^{1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c) : g^+ - g^- \in \tilde{H}^{1/2}(\Gamma_c) \} \quad \text{with norm} \\
\|g\|_\mathcal{X}^2 := \|g^+\|_{H^{1/2}(\Gamma_c)}^2 + \|g^-\|_{H^{1/2}(\Gamma_c)}^2 + \|g^+ - g^-\|_{\tilde{H}^{1/2}(\Gamma_c)}^2.
\]

Equivalently, we define the Hilbert space for Neumann data:

\[
\mathcal{Y} := \{ \varphi = (\varphi^+, \varphi^-) \in H^{-1/2}(\Gamma_c) \times H^{-1/2}(\Gamma_c) : \varphi^+ - \varphi^- \in \tilde{H}^{-1/2}(\Gamma_c) \} 
\]

with similar norm. The Dirichlet problem we consider is: for \( g \in \mathcal{X} \), find \( u \in W^{1,1}(\Omega) \) such that:

\[
\begin{align*}
-\Delta u &= 0, & x \in \Omega, \\
\frac{\partial u}{\partial \nu} + \gamma_c^+ &= g^+, & x \in \Gamma_c. 
\end{align*}
\]
Proposition 1. If $g \in X$, then problem (5) has a unique solution in $W^{1,-1}(\Omega)$.

The problem can be split in the following way. To any function $u$ in $W^{1,-1}(\Omega)$, one associates restrictions $u^\pm$ on $\pi_\pm$ belonging to $W^{1,-1}(\pi_\pm)$. Denote by $\tilde{u}^\pm \in W^{1,-1}(\mathbb{R}^2)$ the mirror reflection of $u^\pm$ over $\pi_\pm$. Then, one can introduce average and jump solutions as

$$
\begin{align*}
\bar{u}_{avg} &= \frac{1}{2}(\tilde{u}^+ + \tilde{u}^-), \\
\bar{u}_{jmp} &= \frac{1}{2}(\tilde{u}^+ - \tilde{u}^-),
\end{align*}
$$

associated to the data

$$
\begin{align*}
\bar{g}_{avg} &= \frac{1}{2}(g^+ + g^-), \\
\bar{g}_{jmp} &= \frac{1}{2}(g^+ - g^-).
\end{align*}
$$

(6)

Normal traces can be similarly decomposed. Due to the convened orientation of the normal on $\Gamma_c$, they take the form:

$$
\begin{align*}
\gamma_c \partial_n \bar{u}_{avg} &= \frac{1}{2} \mathbf{n} \cdot \nabla (\tilde{u}^+ - \tilde{u}^-), \\
\gamma_c \partial_n \bar{u}_{jmp} &= \frac{1}{2} \mathbf{n} \cdot \nabla (\tilde{u}^+ + \tilde{u}^-),
\end{align*}
$$

corresponding to

$$
\begin{align*}
\bar{u}_{avg}, \\
\bar{u}_{jmp},
\end{align*}
$$

respectively, and we have the associated Green' formula:

$$
(\nabla u, \nabla v)_{\Omega} = (\gamma_c \partial_n \bar{u}_{avg}, \gamma_c v_{avg})_{H^{1/2}(\Gamma_c)} + (\gamma_c \partial_n \bar{u}_{jmp}, \gamma_c v_{jmp})_{\tilde{H}^{1/2}(\Gamma_c)},
$$

(8)

since $(\nabla \bar{u}_{avg}, \nabla \bar{u}_{jmp})_{\Omega} = 0$, for $v \in W^{1,-1}(\mathbb{R}^2)$ also decomposed in average and jump parts.

Proposition 2. The solution of the Dirichlet isotropic problem (5), is such that its Neumann trace at $\Gamma_c$ belongs to the space $Y$. There exists a unique application $D : X \to Y$ relating Dirichlet traces to Neumann traces (Dirichlet-to-Neumann map orDtN). Moreover, the energy inequality holds

$$
\langle D g, g \rangle_{\Gamma_c} \geq C \| g \|^2_{X},
$$

(9)

for $g \in X$ and where the vector duality product is given by:

$$
\langle D g, g \rangle_{\Gamma_c} = \langle D \bar{g}_{avg}, \bar{g}_{avg} \rangle_{H^{1/2}(\Gamma_c)} + \langle D \bar{g}_{jmp}, \bar{g}_{jmp} \rangle_{\tilde{H}^{1/2}(\Gamma_c)}.
$$

(10)

Corollary 3. If $g^\pm =: g \in H^{1/2}(\Gamma_c) \setminus C$, the corresponding solution of (5) in $\Omega$ is symmetric with respect to $\Gamma$. Moreover, there exists a unique DtN operator $D_s : H^{1/2}(\Gamma_c) \setminus C \to \tilde{H}^{-1/2}(\Gamma_c)$, and the energy inequality holds

$$
\langle D_s g, g \rangle_{\Gamma_c} \geq C \| g \|^2_{H^{1/2}(\Gamma_c)}.
$$

(11)

Corollary 4. If $g^\pm = \pm g \in \tilde{H}^{1/2}(\Gamma_c)$, the associated solution of (5) is antisymmetric with respect to $\Gamma$. Furthermore, there exists a unique DtN operator $D_{as} : H^{1/2}(\Gamma_c) \to H^{-1/2}(\Gamma_c)$ satisfying

$$
\langle D_{as} g, g \rangle_{\Gamma_c} \geq C \| g \|^2_{H^{1/2}(\Gamma_c)}.
$$

(12)

1.3. Neumann problems

Symmetric and antisymmetric Neumann problems can be stated as follows: find $u_s, u_{as} \in W^{1,-1}(\mathbb{R}^2)$ such that

$$
\begin{align*}
\begin{cases}
-\Delta u_s = 0, & x \in \Omega, \\
\gamma_c \partial_n u_s = \phi, & x \in \Gamma_c,
\end{cases}
\end{align*}
$$

and

$$
\begin{align*}
\begin{cases}
-\Delta u_{as} = 0, & x \in \Omega, \\
\gamma_c^\pm \partial_n u_{as} = \phi, & x \in \Gamma_c,
\end{cases}
\end{align*}
$$

(13)

for data $\phi$ in the space $\tilde{H}^{-1/2}(\Gamma_c)$ and $\phi$ in $H^{-1/2}(\Gamma_c)$. We will refer to the inverse maps as Neumann-to-Dirichlet (NtD) maps.

Proposition 5. The symmetric Neumann problem (13) has a unique solution in $W^{1,-1}(\mathbb{R}^2)/C$ if and only if $\phi \in \tilde{H}^{-1/2}(\Gamma_c)$. Thus, there exists a unique continuous and invertible NtD, denoted by $N_s : \tilde{H}^{-1/2}(\Gamma_c) \to H^{1/2}(\Gamma_c)/C$ satisfying

$$
\langle N_s \phi, \phi \rangle_{\Gamma_c} \geq C \| \phi \|^2_{\tilde{H}^{-1/2}(\Gamma_c)}.
$$

(14)

The inverse of this application is the operator $D_s$ defined in Corollary 3.
**Proposition 6.** The antisymmetric Neumann problem (13) has a unique solution in $W^{1,-1}((\mathbb{R}^2)/\mathbb{C}$ if and only if $\phi \in H^{-1/2}(\Gamma_c)$. Hence, there exists a unique continuous and invertible $\mathcal{N}_{\upsilon}$ from $H^{-1/2}(\Gamma_c) \to \widetilde{H}^{1/2}(\Gamma_c)$, and the energy inequality holds

$$\langle \mathcal{N}_{\upsilon} \phi, \phi \rangle_{\Gamma_c} \geq C \|\phi\|^2_{H^{-1/2}(\Gamma_c)},$$

(15)

The inverse of this application is the operator $\mathcal{D}_{\upsilon}$ defined in Corollary 4.

**2. Main results**

Introduce the following integral logarithmic operators for $x \in I$:

$$\mathcal{L}_1 \varphi(x) := \int_{I} \frac{1}{|x-y|} \varphi(x) \, dx \quad \text{and} \quad \mathcal{L}_2 \varphi(x) := \int_{I} \frac{M(x, y)}{|x-y|} \varphi(x) \, dx,$$

(16)

where the first one is the standard weakly singular single layer operator and where

$$M(x, y) := \frac{1}{2} \left( (y-x)^2 + (w(x) + w(y))^2 \right), \quad \text{with } w(x) := \sqrt{1 - x^2}, \ x \in I.$$

(17)

Lastly, introduce the subspace $H^s_{\pm}((\Gamma_c))$ of functions in $H^{1/2}(\Gamma_c)$ satisfying $\langle g, w \rangle_{\Gamma_c} = 0$.

**2.1. Symmetric problem and the weakly singular operator**

In this case, symmetric Dirichlet and Neumann problems are given via the simple layer potential. For the Neumann version, one just simply introduces the data in the potential whereas for the Dirichlet problem one needs to solve: find $\varphi$ such that

$$\mathcal{L}_1 \varphi(x) = g(x), \quad x \in I,$$

(18)

This integral equation admits an explicit inverse and variational formulations for the equation as well as for its inverse are given in the following proposition:

**Proposition 7.** The symmetric variational formulation of the integral equation (18) in the Hilbert space $\widetilde{H}_0^{-1/2}(\Gamma_c)$ is

$$\langle \mathcal{L}_1 \varphi, \varphi' \rangle_{\Gamma_c} = \langle g, \varphi' \rangle_{\Gamma_c}, \quad \forall \varphi' \in \widetilde{H}_0^{-1/2}(\Gamma_c),$$

(19)

and the associated bilinear form is coercive. The associated operator is $\mathcal{N}_{\upsilon}$ which is a bijection between $\widetilde{H}_0^{-1/2}(\Gamma_c)$ and $H^s_{\pm}(\Gamma_c)$. The inverse operator is bijective from $H^s_{\pm}(\Gamma_c)$ onto $\widetilde{H}_0^{-1/2}(\Gamma_c)$ and is associated to the operator $\mathcal{D}_{\upsilon}$ which is symmetric and coercive in the space $H^s_{\pm}(\Gamma_c)$. It admits two variational formulations:

$$\frac{1}{\pi^2} \langle \mathcal{L}_2 g', (g')' \rangle_{\Gamma_c} = \langle \varphi, g' \rangle_{\Gamma_c}, \quad \forall g' \in H^s_{\pm}(\Gamma_c),$$

(20)

which gives a first norm on the space $H^s_{\pm}(\Gamma_c)$:

$$\frac{1}{\pi^2} \langle \mathcal{L}_2 g', (g')' \rangle_{\Gamma_c} \geq C \|g\|^2_{H^s_{\pm}(\Gamma_c)}, \quad \forall g \in H^s_{\pm}(\Gamma_c).$$

(21)

The second one is

$$\frac{1}{2\pi^2} \int_{\Gamma_c} \frac{1}{|x-y|} \log \left[ \frac{M(x,y)}{|x-y|} \right] (g(x)-g(y))(g'(x)-g'(y)) \, dy \, dx = \langle \varphi, g' \rangle_{\Gamma_c},$$

(22)

for all $g' \in H^s_{\pm}(\Gamma_c)$, and we obtain a second norm on the space $H^s_{\pm}(\Gamma_c)$ which is:

$$\int_{\Gamma_c} \frac{1}{w(x)w(y)} \frac{(g(x)-g(y))^2}{(x-y)^2} \, dy \, dx \geq C \|g\|^2_{H^s_{\pm}(\Gamma_c)}, \quad \forall g \in H^s_{\pm}(\Gamma_c).$$

(23)

Although the Dirichlet problem (5) admits a unique solution for all $g = g^0$ in $H^{1/2}(\Gamma_c)$, the solution to a constant data, e.g. corresponding to $\psi^0 \equiv 1$, is such that $\varphi = 0$. Thus, the integral representation (18) cannot describe this constant solution. The exact image by the operator $\mathcal{N}_{\upsilon}$ of the space $\widetilde{H}_0^{-1/2}(\Gamma_c)$ is the subspace $H^s_{\pm}(\Gamma_c)$ which also does not contain the trace of the constant function.
2.2. Antisymmetric problem and the hypersingular operator

The solution for the antisymmetric Dirichlet problem is retrieved by direct action of the double layer potential. However, for the Neumann version, one must first solve the hypersingular integral equation for $\alpha$ (the jump of the Dirichlet trace):

$$\varphi(x) = \int_I \frac{1}{|x - y|^2} \alpha(y) \, dy, \quad \text{for } x \in I. \quad (24)$$

**Theorem 8.** A symmetric variational formulation for (24) in the Hilbert space $\tilde{H}^{1/2}(\Gamma_c)$ is given by

$$\langle L_1 \alpha', (\alpha')^T \rangle_{\Gamma_c} = \langle \varphi, \alpha \rangle_{\Gamma_c}, \quad \forall \alpha' \in \tilde{H}^{1/2}(\Gamma_c),$$

which is coercive. The associated operator $D_{\text{as}}$ is a bijection from $\tilde{H}^{1/2}(\Gamma_c)$ to $H^{-1/2}(\Gamma_c)$. This operator admits an alternative variational formulation:

$$\iint_I \frac{(x - y)(\alpha'(x) - \alpha'(y))}{|x - y|^2} \, dx \, dy + \iint_I \frac{\alpha(x)\alpha'(x)}{1 - x^2} \, dx = \langle \varphi, \alpha \rangle_{\Gamma_c},$$

for all $\alpha' \in \tilde{H}^{1/2}(\Gamma_c)$, and the next expression is a norm on $\tilde{H}^{1/2}(\Gamma_c)$:

$$\iint_I \frac{(x - y)^2}{|x - y|^2} \, dx \, dy + \iint_I \frac{\alpha(x)^2}{1 - x^2} \, dx \geq C \|\alpha\|_{\tilde{H}^{1/2}(\Gamma_c)}^2, \quad \forall \alpha \in \tilde{H}^{1/2}(\Gamma_c).$$

The inverse operator is associated to the operator $N_{\text{as}}^{-1} = D_{\text{as}}$, and it is a bijection of $H^{-1/2}(\Gamma_c)$ onto $\tilde{H}^{1/2}(\Gamma_c)$, symmetric and coercive in the space $H^{-1/2}(\Gamma_c)$. It admits the following variational formulation:

$$\frac{1}{\pi^2} \langle L_2 \varphi, (\varphi')^T \rangle_{\Gamma_c} = \langle \alpha, \varphi \rangle_{\Gamma_c}, \quad \forall \varphi \in H^{-1/2}(\Gamma_c), \quad (28)$$

and thus, the following expression is a norm on the space $H^{-1/2}(\Gamma_c)$:

$$\langle L_2 \varphi, \varphi \rangle_{H^{-1/2}(\Gamma_c)} \geq \frac{C}{\pi^2} \|\varphi\|_{H^{-1/2}(\Gamma_c)}^2, \quad \forall \varphi \in H^{-1/2}(\Gamma_c).$$

**Proposition 9.** The subspace $\tilde{H}^{1/2}(\Gamma_c)$ is exactly the space of functions $g$ in the space $H_*^{1/2}(\Gamma_c)$ such that $w^{-1} g$ is in $L^2(\Gamma_c)$. The space $\tilde{H}^{1/2}(\Gamma_c)$ is exactly the subspace of $H^{-1/2}(\Gamma_c)$, orthogonal in the duality product with $\tilde{H}^{1/2}(\Gamma_c)$ to the space of functions $w \varphi$, where $\varphi$ varies in $L^2(\Gamma_c)$.

2.3. Calderón-type identities

Two derivation operators have appeared in the above propositions, one whose domain lies on $\tilde{H}^{1/2}(\Gamma_c)$ and another acting on $H_*^{1/2}(\Gamma_c)$. Since $\tilde{H}^{1/2}(\Gamma_c)$ can be extended by zero to be a subspace of $H^{1/2}(\mathbb{R})$ which is a subspace of the distribution space $S'\mathbb{R}$, the first derivation operator, denoted by $D$, is defined distributionally. We will denote the second one as $-D^*$ taken in classical sense.

**Proposition 10.** The derivation operator $D$ is continuous and surjective from the space $\tilde{H}^{1/2}(\Gamma_c)$ onto $\tilde{H}^{-1/2}(\Gamma_c)$, while the derivation operator $-D^*$ is continuous and surjective from the space $H_*^{1/2}(\Gamma_c)$ onto the space $H^{-1/2}(\Gamma_c)$. Moreover, the operator $D^*$ is the adjoint of the operator $D$ with respect to the duality product in $L^2(\Gamma_c)$.

Finally, one can prove some properties linking these derivation operators $D$ and $D^*$ and the logarithmic operators previously introduced.

**Proposition 11.** The operators $L_1$, $L_2$, $D$, $D^*$ are linked by the identities

$$- L_2 \circ D^* \circ L_1 \circ D = \text{Id}_{\tilde{H}^{1/2}(\Gamma_c)}, \quad - L_1 \circ D \circ L_2 \circ D^* = \text{Id}_{H_*^{1/2}(\Gamma_c)},$$

$$- D \circ L_2 \circ D^* \circ L_1 = \text{Id}_{\tilde{H}^{-1/2}(\Gamma_c)}, \quad - D^* \circ L_2 \circ D \circ L_1 = \text{Id}_{H^{-1/2}(\Gamma_c)}.$$
References