Mathematical Problems in Mechanics

A nonlinear Korn inequality with boundary conditions and its relation to the existence of minimizers in nonlinear elasticity

Une inégalité de Korn non linéaire et son relation à l’existence de minimiseurs en elasticité non linéaire

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1. Introduction and main results

We use the following notation. \( \mathbb{M}_n, \mathbb{S}_n, \mathbb{M}_n^+, \) and \( \mathbb{O}_n^+ \) denote respectively the space of square matrices of order \( n \), the space of all symmetric matrices of order \( n \), the set of all matrices \( A \in \mathbb{M}_n^+ \) such that \( \det A > 0 \), and the set of all matrices \( A \in \mathbb{M}_n^+ \) such that \( A^T A = I \). \( A^T \) and \( I \) denote respectively the transpose of the matrix \( A \) and the identity matrix in \( \mathbb{M}_n^+ \). The inner products in \( \mathbb{R}^n \) and \( \mathbb{M}_n^+ \) are respectively denoted \( \cdot \) and \( : \). The Euclidean norm in \( \mathbb{R}^n \) and the Frobenius norm in \( \mathbb{M}_n^+ \) are both denoted by \( | \cdot | \). Vector and matrix fields are denoted by boldface letters. The gradient of a vector field \( \mathbf{u} \) is the matrix field \( \nabla \mathbf{u} \), whose columns are the partial derivatives of \( \mathbf{u} \). The divergence of a matrix field \( \mathbf{A} \) is the vector field \( \text{div} \mathbf{A} \), whose components are the divergences of the row vectors of \( \mathbf{A} \). The matrix field \( \mathbf{I} \) is defined by \( \mathbf{I}(x) = I \) for all
\( x \in \mathbb{R}^n, L^p(\Omega), W^{k,p}(\Omega) \) or \( H^k(\Omega) \) if \( p = 2 \), and \( C^k(\Omega) \) denote respectively Lebesgue spaces, Sobolev spaces, and \( k \) times continuously differentiable functions.

The first result of this Note is the following nonlinear Korn inequality with boundary conditions:

**Theorem 1.** Let \( \Omega \) be a bounded connected open subset of \( \mathbb{R}^3 \) with a Lipschitz-continuous boundary \( \partial \Omega \) and let \( \lambda < 1 \). Then there exists a constant \( C \) such that

\[
\|\nabla u - \nabla v\|_{L^2(\Omega)} \leq \frac{C}{(1 - \lambda)^{2+n/2}} \|\nabla u^T \nabla u - \nabla v^T \nabla v\|_{L^2(\Omega)}
\]

for all pairs \( (u, v) \in W^{1,4}(\Omega; \mathbb{R}^3) \times C^1(\overline{\Omega}; \mathbb{R}^3) \) that satisfy \( u(x) = v(x) \) for almost all \( x \in \partial \Omega \), \( \det(\nabla u(x)) > 0 \) for almost all \( x \in \Omega \), \( v \) is injective, and \( \|\nabla v - I\|_{L^2(\Omega)} \leq \lambda \).

Note that the above inequality remains valid for deformation fields that are possibly non-injective and satisfy more general boundary conditions, as will be shown elsewhere.

The inequality of Theorem 1 is reminiscent of another inequality, established earlier by Ciarlet and Mardare [4], showing that, given any vector field \( v \in C^1(\overline{\Omega}; \mathbb{R}^3) \) satisfying \( \det(\nabla v) > 0 \) in \( \overline{\Omega} \), there exists a constant \( C(v) \) such that

\[
\inf_{\tilde{R} \in C^0_\text{ad}} \|\nabla u - R \nabla v\|_{L^2(\Omega)} \leq C(v) \|\nabla u^T \nabla u - \nabla v^T \nabla v\|_{L^1(\Omega)}^{1/2}
\]

for all \( u \in H^1(\Omega; \mathbb{R}^3) \) satisfying \( \det(\nabla u) > 0 \) a.e. in \( \Omega \). The idea that the exponent \( 1/2 \) can be dropped at the expense of a stronger norm in the right-hand side of the last inequality is due to Blanchard [2].

Theorem 1 has applications in nonlinear elasticity. Consider a body with reference configuration \( \Omega \subset \mathbb{R}^3 \), made of a hyperelastic material characterized by a stored energy function \( \tilde{W} : \Omega \times M^3_+ \to \mathbb{R} \), and subjected to body forces of density \( f : \Omega \to \mathbb{R}^3 \). Then the total energy corresponding to a deformation \( u : \Omega \to \mathbb{R}^3 \) of the body is given by

\[
J(u) = \int_\Omega \tilde{W}(x, \nabla(u(x))) \, dx - \int_\Omega f(x) \cdot u(x) \, dx.
\]

If \( \tilde{W} \) is polyconvex and satisfies suitably growth conditions, then \( J \) possesses a minimizer over a suitable set of admissible deformations \( u \); cf. Ball [1]. If in addition the Euler–Lagrange equation corresponding to this minimization problem has a solution by the implicit function theorem, then this solution coincides with the minimizer above; cf. Zhang [8].

The second result of this Note is that the total energy \( J \) possesses a minimizer over a suitable set of admissible deformations \( u \) for some hyperelastic materials that does not necessarily meet the assumptions of Ball and Zhang. We assume that the stored energy function \( \tilde{W} \) satisfies the axiom of material frame-indifference, so that there exists a function \( W : \Omega \times S^3 \to \mathbb{R} \) such that

\[
\tilde{W}(x, F) = W(x, E), \quad E = \frac{1}{2}(F^T F - I)
\]

for almost all \( x \in \Omega \) and for all \( F \in M^3_+ \).

We recall that \( \Omega \) is a natural configuration of the body if \( \frac{\partial W}{\partial E}(x, 0) = 0 \) for all \( x \in \Omega \).

**Theorem 2.** Let \( \Omega \) be a bounded connected open subset of \( \mathbb{R}^3 \) with a boundary of class \( C^2 \) and let \( f \in L^p(\Omega; \mathbb{R}^3) \), \( p > 3 \). Assume that \( W \in C^3(\overline{\Omega} \times S^3) \), that \( \frac{\partial W}{\partial E}(x, 0) = 0 \) for all \( x \in \Omega \), and that there exist constants \( \alpha > 0 \) and \( \varepsilon > 0 \) such that

\[
W(x, E + H) \geq W(x, E) + \frac{\partial W}{\partial E}(x, E) : H + \alpha |H|^2 \quad \text{for all } x \in \overline{\Omega}, \ E \in S^3, \ H \in S^3, \ |E| < \varepsilon.
\]

Then there exists a constant \( \delta > 0 \) with the following property: If \( \|f\|_{L^p(\Omega)} < \delta \), then the functional \( J \) defined by Eqs. (1)–(2) has a unique minimizer over the set of admissible deformations defined by

\[
\mathcal{M} = \{ u \in W^{1,4}(\Omega; \mathbb{R}^3); \ \det(\nabla u) > 0 \ \text{a.e. in } \Omega, \ u(x) = x \ \text{for all } x \in \partial \Omega \}.
\]

Moreover, the minimizer \( v \) belongs to the space \( C^1(\overline{\Omega}; \mathbb{R}^3) \), is injective from \( \overline{\Omega} \) into \( \mathbb{R}^3 \), and satisfies \( \det(\nabla v(x)) > 0 \) for all \( x \in \overline{\Omega} \).

Theorem 2 applies in particular to the stored energy function of Saint Venant–Kirchhoff materials, given in terms of its Lamé constants \( \lambda > 0 \) and \( \mu > 0 \) by the expression

\[
\tilde{W}(x, F) = W(x, E) = \frac{\lambda}{2} (\text{tr} E)^2 + \mu |E|^2, \quad E = \frac{1}{2}(F^T F - I), \quad \text{for all } (x, F) \in \Omega \times M^3_+.
\]

Note that the previous existence theorems of Ball [1], Zhang [8], and Ciarlet and Mardare [5] do not apply in that case.
Theorem 2 still holds if \( W \) is of class \( C^3 \) only over the set \( \Omega \times [E \in \mathbb{R}^3; \quad |E| < \varepsilon] \) for some \( \varepsilon > 0 \). Note also that the assumptions of Theorem 2 imply that the mapping \( E \mapsto W(x, E) \) is convex in a neighborhood of the zero matrix. But this does not imply that the mapping \( F \mapsto W(x, F) \) is convex in a neighborhood of the identity matrix (\( W \) and \( W \) are related by (2)), so the direct methods in the calculus of variations cannot be used to prove Theorem 2.

2. Proof of Theorem 1

Extend the mapping \( v \) to an open ball \( B \subset \mathbb{R}^n \) containing \( \Omega \) and let \( u(x) = v(x) \) for all \( x \in B \setminus \Omega \). Define the composite mapping \( \varphi := u \circ v^{-1} \) and note that \( \varphi \in W^{1,4}(B; \mathbb{R}^n) \) and \( \det(\nabla \varphi(x)) > 0 \) for almost all \( x \in B \); cf. Clariet [3, Theorems 5.5.1 and 5.5.2].

The geometric rigidity lemma of Friesecke, James and Müller [6, Theorem 3.1] implies that there exists a constant \( K \) independent of \( u \) and \( v \) such that

\[
\inf_{K \in O_n} \int_B |\nabla \varphi(x) - R|^2 \, dx \leq K \int_B |\nabla \varphi(x) - Q|^2 \, dx.
\]

The identity \( I = \frac{1}{|B\setminus\Omega|} \int_{B\setminus\Omega} \nabla \varphi(x) \, dx \), where \( |B \setminus \Omega| \) denotes the Lebesque measure of the set \( B \setminus \Omega \), implies that

\[
\|\nabla \varphi - I\|_{L^2(B)} \leq \left( \int_{B\setminus\Omega} |\nabla \varphi(x) - R|^2 \, dx \right)^{1/2} + \frac{|B|^{1/2}}{|B \setminus \Omega|} \int_{B \setminus \Omega} |R - \nabla \varphi(x)| \, dx \quad \text{for all } R \in O^+_n.
\]

Combining these inequalities gives

\[
\|\nabla \varphi - I\|_{L^2(B)} \leq C_0 \left( \int_B \inf_{Q \in O^+_n} |\nabla \varphi(x) - Q|^2 \, dx \right)^{1/2}, \quad \text{where } C_0 = K^{1/2} \left( 1 + \frac{|B|^{1/2}}{|B \setminus \Omega|^{1/2}} \right),
\]

which next implies, by using that \( \inf_{Q \in O^+_n} |F - Q|^2 \leq |F^T F - I|^2 \) for all \( F \in M^+_n \), that

\[
\|\nabla \varphi - I\|_{L^2(\Omega)} = \|\nabla \varphi - I\|_{L^2(B)} \leq C_0 \|\nabla \nabla^T \nabla \varphi - I\|_{L^2(\Omega)} = C_0 \|\nabla \varphi^T \nabla \varphi - I\|_{L^2(\Omega)}.
\]

Now, by changing variables in the integrals defining the \( L^2(\Omega) \)-norms above, we obtain that

\[
\int_\Omega |\nabla u - v|^2 \, dy \leq C_0 \int_\Omega |\nabla u^T \nabla u - v^T \nabla v|^2 |(\nabla v)^{-1}|^4 \det(\nabla v) \, dy.
\]

Since \( |\nabla v(y) - I| \leq \lambda < 1 \) and \( |\nabla v^{-1}(y) - I| \leq \frac{1}{1-\lambda} \) for all \( y \in \Omega \), we deduce that the singular values of the matrix \( \nabla v(y) \) are contained in the interval \( \left[ \frac{1-\lambda}{1+\lambda}, \frac{1+\lambda}{1-\lambda} \right] \). Therefore there exist constants \( C_1 > 0 \) and \( C_2 < \infty \) such that \( \frac{\det(\nabla v)}{|\nabla v|^2} \geq C_1 (1-\lambda)^n \) and \( |(\nabla v)^{-1}|^4 \det(\nabla v) \leq C_2 (1-\lambda)^{-n} \). Combined with the previous inequality, this implies that

\[
\|\nabla u - v\|_{L^2(\Omega)} \leq C_0(C_2/C_1)^{1/2} \left( \frac{1}{1-\lambda} \right)^{n/2} \|\nabla u^T \nabla u - v^T \nabla v\|_{L^2(\Omega)}.
\]

3. Proof of Theorem 2

The assumptions on the function \( W \) imply that \( J(u) \in \mathbb{R} \cup \{+\infty\} \) is well defined. They also imply that the Euler–Lagrange equation formally derived from the total energy \( J: \mathcal{M} \to \mathbb{R} \cup \{+\infty\} \), that is

\[
-\operatorname{div} \left( \nabla v \frac{\partial W}{\partial E} (\cdot, \mathbf{E}(v)) \right) = f \quad \text{in } \Omega \quad \text{and} \quad v(x) = x \quad \text{for all } x \in \Gamma,
\]

has a solution \( v = v(t) \in W^{2,p}(\Omega; \mathbb{R}^3) \) given by the implicit function theorem for any \( f \in L^p(\Omega; \mathbb{R}^3) \) such that \( \|f\|_{L^p(\Omega)} < \delta \); cf., e.g., Quintela-Estévez [7, Theorem 4.5]. Moreover, \( v \) is of class \( C^1 \) and injective from \( \Omega \) into \( \mathbb{R}^3 \) (cf. Ciarlet [3, Theorems 5.5.1 and 5.5.2]) and

\[
\lambda(\delta) := \sup \left\{ \|\nabla v(f) - I\|_{L^\infty(\Omega)} ; \ f \in L^p(\Omega; \mathbb{R}^3), \ \|f\|_{L^p(\Omega)} < \delta \right\} \to 0 \quad \text{as } \delta \to 0.
\]

Choosing \( \delta \) sufficiently small, we have \( \|E(v)\|_{L^\infty(\Omega)} \leq 3 \|\nabla v - I\|_{L^\infty(\Omega)} < 3\lambda(\delta) < \varepsilon \). Then

\[
J(u) - J(v) \geq \int_{\Omega} \left( \frac{\partial W}{\partial E} (\cdot, \mathbf{E}(v)) : (\mathbf{E}(u) - \mathbf{E}(v)) + \alpha |\mathbf{E}(u) - \mathbf{E}(v)|^2 \right) \, dx - \int_{\Omega} f \cdot (u - v) \, dx.
\]
But
\[
\frac{\partial W}{\partial E}(\cdot, E(v)) : (E(u) - E(v)) = \frac{\partial W}{\partial E}(\cdot, E(v)) : \nabla v^T \nabla (u - v) + \frac{\partial W}{\partial E}(\cdot, E(v)) : \frac{\nabla (u - v)^T \nabla (u - v)}{2}
\]
(since \(\frac{\partial W}{\partial E}(x, E(v)(x)) \in S^3\) for all \(x \in \Omega\)) and
\[
\int_\Omega \frac{\partial W}{\partial E}(\cdot, E(v)) : \nabla v^T \nabla (u - v) \, dx = - \int_\Omega \text{div} \left( \nabla v \frac{\partial W}{\partial E}(\cdot, E(v)) \right) : (u - v) \, dx.
\]
Hence
\[
J(u) - J(v) \geq \int_\Omega \frac{\partial W}{\partial E}(\cdot, E(v)) : \frac{(\nabla u - \nabla v)^T (\nabla u - \nabla v)}{2} \, dx + \alpha \int_\Omega \|E(u) - E(v)\|^2 \, dx
\]
\[
\geq \frac{\alpha}{4} \|\nabla u^T \nabla u - \nabla v^T \nabla v\|^2_{L^2(\Omega)} - \frac{1}{2} \left\| \frac{\partial W}{\partial E}(\cdot, E(v)) \right\|_{L^\infty(\Omega)}^2 \|\nabla u - \nabla v\|^2_{L^2(\Omega)}.
\]
Then the nonlinear Korn inequality of Theorem 1 shows that, for some constant \(C\),
\[
J(u) - J(v) \geq \left( \frac{\alpha(1 - \lambda(\delta))^2}{4C^2} - \frac{1}{2} \left\| \frac{\partial W}{\partial E}(\cdot, E(v)) \right\|_{L^\infty(\Omega)}^2 \right) \|\nabla u - \nabla v\|^2_{L^2(\Omega)}.
\]
But \(\lim_{\delta \to 0} \frac{\alpha(1 - \lambda(\delta))^2}{4C^2} - \frac{1}{2} \left\| \frac{\partial W}{\partial E}(\cdot, E(v)) \right\|_{L^\infty(\Omega)}^2 = \frac{\alpha}{4C^2}\). Hence \(J(u) - J(v) > 0\) for all \(u \in M, u \neq v\), if \(\delta\) is chosen small enough.

References