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# Asymptotically conformal similarity between Julia and Mandelbrot sets ☆

## *Similitude conforme asymptotiquement entre les ensembles de Julia et de Mandelbrot*

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ABSTRACT

An almost conformal local similarity between the connectedness locus $\mathcal{M}_d$ and the corresponding Julia set is true for almost every point of $\partial\mathcal{M}_d$ with respect to harmonic measure. The harmonic measure is supported on Lebesgue density points of the complement of $\mathcal{M}_d$ which are not accessible from outside within John angles and at which the boundary of $\mathcal{M}_d$ spirals infinitely often in both directions. A more general result can be obtained for $d = 2$ in terms of the renormalization property. Finally, we prove that for almost all $c \in \partial\mathcal{M}_d$ in the sense of harmonic measure the Lyapunov exponent of $c$ under iterates of $z^d + c$ is equal to $\log d$.


RÉSUMÉ

En presque tout point $c$ par rapport à la mesure harmonique, le lieu de connectivité $\mathcal{M}_d$ est asymptotiquement similaire au sens conforme à l'ensemble de Julia $\mathcal{J}_c$ près de $c$. Tout point de concentration de la mesure harmonique est un point de densité du complémentaire de $\mathcal{M}_d$ qui n'est pas bien accessibles du complémentaire de $\mathcal{M}_d$, autour duquel la frontière de $\mathcal{M}_d$ est en spirale une infinité de fois dans deux directions opposées. Pour l'ensemble de Mandelbrot ($d = 2$) on peut obtenir un résultat plus général en terme de la propriété de renormalisation. Finalement, on démontre que pour presque toute valeur de $c \in \partial\mathcal{M}_d$ par rapport à la mesure harmonique, l'exposant de Lyapunov de $c$ sous la dynamique de $z^d + c$ est égal à $\log d$.


### Version française abrégée

Soit $f_c(z) = z^d + c$, $z, c \in \mathbb{C}$ et $d \geqslant 2$ un entier. Alors son ensemble Julia $\mathcal{J}_c$ est défini comme $\partial\mathcal{K}_c$, où $\mathcal{K}_c = \{z \in \mathbb{C}\colon \sup_{n>0} |f_c^n(z)| < \infty\}$ est un ensemble de Julia rempli. L'ensemble $\mathcal{M}_d$ (lieu de connexité des ensembles de Julia $\mathcal{J}_c$) est l'ensemble des paramètres $c$ pour lesquels les ensembles correspondants de Julia $\mathcal{J}_c$ sont connexes. Pour les polynômes unicritiques $f_c(z) = z^d + c$, la condition de Collet–Eckmann est que

$$\liminf_{n\to\infty} \frac{\log |Df_c^n(c)|}{n} > 0.$$

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L'étude de la frontière de l'ensemble $\mathcal{M}_d$ est basée souvent sur la géométrie et la combinatoire des partitions de Yoccoz, les mouvements holomorphes et des méthodes de probabilité. Décrire la frontière de $\mathcal{M}_d$ en *tout* point est un problème bien connu en dynamique complexe, mais c'est aussi un problème qui est très difficile.

Dans [8,15] et [9] un but plus modeste a été poursuivi, i.e. l'étude de la géométrie du lieu de connexité et la dynamique du polynôme correspondant dans son espace de phase en presque tout point au sens de la mesure harmonique sur $\partial\mathcal{M}_d$. Il apparaît qu'en presque tout point $c$, on a deux informations importantes avec lesquelles on peut travailler : le lieu de connexité est localement similaire à l'ensemble de Julia $\mathcal{J}_c$ près de $c$, et la dynamique de $f_c$ montre une certaine hyperbolicité, en particulier $f_c$ satisfait la condition de Collet–Eckmann. La similarité forte mentionnée ici signifie qu'il existe une application quasi-conforme $\Upsilon : \mathbb{C} \mapsto \mathbb{C}$, $\Upsilon(c) = c$, et un continuum $V \supset \mathcal{J}_c$ tels que pour tout disque géométrique $W \ni c$ assez petit dans l'espace de paramètres, la dilatation de $\Upsilon$ sur $W$ tend vers 1 quand $\operatorname{diam} W \to 0$ et $\Upsilon(\mathcal{M}_d \cap W) \subset V$ est « proche » de $J_c \cap \Upsilon(W)$. Le caractère « proche » peut être compris ici dans le sens où la distance de Hausdorff entre ces ensembles est petite comparée au $\operatorname{diam} \Upsilon(W)$ et le rapport tend vers 0 lorsque $\operatorname{diam}(W) \to 0$. On peut aussi le comprendre dans le sens métrique, i.e. que la mesure de la différence symétrique de ces ensembles tend vers 0 comparée à la mesure de $\Upsilon(W)$.

De plus, l'application $\Upsilon$ est conforme sur $W \setminus \Upsilon^{-1}(V)$ et $\operatorname{aire}(W \setminus \Upsilon^{-1}(V))/\operatorname{aire}(W)$ tend vers 0. Par conséquent, $c$ est un point de densité de Lebesgue de $\mathbb{C} \setminus \mathcal{M}_d$ et la correspondance entre $\mathcal{M}_d$ et $\mathcal{J}_c$ est asymptotiquement conforme dans le sens de la limite géométrique. On obtient (Théorème 4) que chaque point de concentration de la mesure harmonique est un point de densité du complémentaire de $\mathcal{M}_d$.

Cette similarité forte entre $\mathcal{M}_d$ et $\mathcal{J}_c$ près de $c$, avec les propriétés d'expansivité de $f_c$ mène à la stratégie suivante d'« amplification » des caractéristiques géométriques. Supposons que $\mathcal{M}_d$ ait une caractéristique géométrique locale en $c$. Alors la similarité quasi-conforme $\Upsilon$ peut nous permettre de transporter cette même caractéristique à l'ensemble de Julia $\mathcal{J}_c$ au point $c$. Ensuite, on utilise l'expansivité de la dynamique de $f_c$ pour « amplifier » cette caractéristique à grande échelle et la distribuer autour de $\mathcal{J}_c$. En passant à la limite, la caractéristique géométrique locale de $\mathcal{J}_c$ en $c$ devient une propriété géométrique globale de $\mathcal{J}_c$. Par exemple, nous prouvons que pour tout $d \geqslant 2$, presque tous les points $c \in \partial\mathcal{M}_d$ au sens de la mesure harmonique, ne sont pas bien accessibles (i.e. ne sont pas accessibles selon un angle de John) à partir du complémentaire de $\mathcal{M}_d$. Supposons donc au contraire qu'il existe un ensemble $Z \subset \mathcal{M}_d$ de mesure harmonique positive tel que tous les $c \in Z$ sont bien accessibles. On invoque le Théorème 3 pour conclure que $c \in \mathcal{J}_c$ est aussi bien accessible à partir du complémentaire de $\mathcal{J}_c$ pour $c \in Z \cap \mathcal{G}$ où $\mathcal{G}$ correspond à une classe des applicactions de Collet–Eckmann introduite en [8]. En utilisant l'expansivité forte de la dynamique de $f_c$, $c \in \mathcal{G}$, on amplifie cette propriété de façon uniforme à grande échelle. On obtient que $\mathcal{J}_c$ est uniformément bien accessible sur un ensemble dense de $\mathcal{J}_c$. En passant à la limite, on obtient que $\mathcal{J}_c$ est un domaine de John et par [3], $c$ est non-récurrent. L'ensemble des $f_c$ avec la valeur critique $c = f_c(0)$ non-récurrent est de mesure harmonique nulle, contradiction.

Pour l'ensemble de Mandelbrot $\mathcal{M}$ (le cas quadratique, $d = 2$) on peut obtenir un résultat plus général que le Théorème 3, qui ne fait pas référence à la mesure harmonique. Notamment, le Théorème 2 affirme que pour tout paramètre $c \in \partial\mathcal{M}$, exceptées les valeurs $c$ pour lesquelles $f_c$ est infiniment renormalisable, il existe une base d'ouverts autour de $c$ telle que pour tout $U$ appartenant à cette base, il y a une application quasi-conforme $\Upsilon_U$, $\Upsilon_U(c) = c$, avec une dilatation uniformément bornée et qui converge vers 1 lorsque $\operatorname{diam} U \to 0$, de sorte que $\Upsilon_U(\mathcal{M} \cap U)$ soit « proche » de $J_c \cap \Upsilon_U(U)$ au même sens que dans le Théorème 3. En général, on ne peut pas améliorer les estimations du Théorème 2 pour obtenir une seule fonction de similarité $\Upsilon$ entre $\mathcal{M}$ et $\mathcal{J}_c$ au voisinage de $c$. Il y a une obstruction topologique ; notamment, il existe $c$ satisfaisant les hypothèses du Théorème 3 et qui est arbitrairement proche du cardioide principal des petites copies de $\mathcal{M}$ pour une infinité d'échelles en $c$. Donc, il existe une base d'ouverts $D$ en $c$, telle que les images de $\mathcal{M} \cap D$ par n'importe quel homéomorphisme quasi-conforme ne peuvent pas être proches de $\mathcal{J}_c$ en $c$, au sens du Théorème 2, puisque $\mathcal{J}_c$ est une dendride, i.e. $\mathcal{J}_c = \partial\mathcal{J}_c$. La preuve du Théorème 2 est fondée sur la décroissance uniforme de la géométrie. Lorsque la multiplicité du point critique est 2, cette propriété est démontrée en détail dans [7].

On ne sait pas si la mesure de Lebesgue de $\partial\mathcal{M}$ est 0 ou non. Shishikura a prouvé que $\operatorname{HD}(\partial\mathcal{M}) = 2$. Le Théorème 1 contribue à une conjecture de longue date affirmant que l'aire de $\partial\mathcal{M}$ est zéro. Notamment, l'ensemble $\{c \in \partial\mathcal{M}\}$, excepté les valeurs $c$ pour lesquels $f_c$ est infiniment renormalisable, a une mesure de Lebesgue 0.

Finalement, on obtient que pour presque toute valeur de $c \in \partial\mathcal{M}_d$ par rapport à la mesure harmonique

$$\lim_{n\to\infty} \frac{\log|Df_c^n(c)|}{n} = \log d.$$

## 1. Julia and Mandelbrot sets

A *filled Julia set* $\mathcal{K}_c$ of a unicritical polynomial $f_c(z) = z^d + c$ is defined as $\mathcal{K}_c = \{z \in \mathbb{C}\colon \sup_{n>0} |f_c^n(z)| < \infty\}$. The connectedness locus $\mathcal{M}_d$ is the set of parameters $c$ for which the corresponding Julia set $\mathcal{J}_c = \partial\mathcal{K}_c$ is connected. A direct argument shows that $\mathcal{M}_d = \{c \in \mathbb{C}\colon \sup_{n>0} |f_c^n(c)| < \infty\}$.

It is well known that $\mathcal{M}_d$ is a full compact, that is its complement is an open topological disk, with non-empty interior. For $c \in \mathcal{M}_d$ the critical orbit $\{f^n(c)\}$ belongs to the filled Julia set $\mathcal{K}_c$. When $c$ traverses $\mathcal{M}_d$ in the outward direction, $\mathcal{K}_c$ which is initially connected bifurcates into a Cantor set outside of $\mathcal{M}_d$. If $f_c$ has an attracting periodic orbit then $c$ belongs to a hyperbolic component of the interior of $\mathcal{M}_d$. Every hyperbolic component is a topological disk with a piecewise analytic boundary. A great deal of work was devoted to the Fatou conjecture [5] which asserts that the interior of $\mathcal{M}_d$ consists of only hyperbolic components. This is still an open problem but it is known that the local connectivity of the boundary would imply the conjecture.

The connectedness locus $\mathcal{M}_2$ for quadratic polynomials is known as the *Mandelbrot set* and is usually denoted by $\mathcal{M}$. Topological aspects of the boundary of $\mathcal{M}$ were addressed in [14]. Typical parameters with respect to the harmonic measure were studied in [8,15]. These two approaches yield very different metric properties of generic Julia sets. For example, Hausdorff dimension of a topologically generic Julia set $\mathcal{J}_c$, $c \in \partial\mathcal{M}$, is equal to 2 while typically, with respect to the harmonic measure, Hausdorff dimension of $\mathcal{J}_c$ is strictly less than 2.

In [9], the distribution of the planar Lebesgue and the harmonic measures on $\mathcal{M}_d$ are studied in details. The objective of this paper is to formulate and explain the main theorems proven in [9]. We will give also a complete proof of Theorem 5 and an outline of the proof of some parts of Theorem 4.

## 2. Mandelbrot set and weak similarity

Generally, it is not known if Lebesgue measure of $\partial\mathcal{M}$ is 0 or not. In [14], it was proved that $\mathrm{HD}(\partial\mathcal{M}) = 2$. In fact, the proof exploits only Misiurewicz parameters ($c$ is not recurrent) which are shown to have Hausdorff dimension 2. The set of Misiurewicz parameters is of harmonic and Lebesgue measure 0, see [8,15].

We say that a point $x$ is a *Lebesgue weak density point* of a set $S \subset \mathbb{C}$ if

$$\limsup_{r\to 0} \frac{\operatorname{area} S \cap D(x,r)}{\operatorname{area} D(x,r)} = 1. \tag{1}$$

If additionally the limit in (1) exists then $x$ is named a Lebesgue density point of $S$.

A self-similar structure of the Mandelbrot set around certain parameters of its boundary was first observed numerically. It was explained qualitatively by the concept of tuning introduced in [4].

**Definition 2.1.** A unicritical polynomial $f_c(z) = z^d + c$ is called *renormalizable* with period $n$ if there exist topological disks $V$ and $U$ so that $\overline{U} \subset V$, $f^n : U \to V$ is proper holomorphic with degree $d$ and $f^{kn}(0) \in U$ for any non-negative integer $k$. If $f_c(z)$ is renormalizable with infinitely many different periods then $f_c(z)$, or simply $c$, is called infinitely renormalizable.

We will describe the part of the boundary of $\mathcal{M}$ which corresponds to polynomials which are not infinitely renormalizable. Theorem 1 contributes towards the conjecture that the area of $\partial\mathcal{M}$ is zero, see [10], and improves the theorem of Shishikura announced in [13] that the set $\{c \in \partial\mathcal{M}\}$ excepting those values for which $f_c$ is infinitely renormalizable has planar Lebesgue measure 0.

A parameter $c \in \mathcal{M}_d$ is named recurrent if $c$ is recurrent for $f_c(z)$.

**Theorem 1.** *Every parameter $c \in \mathcal{M}$ which is recurrent but not infinitely renormalizable and does not lie on the boundary of any hyperbolic component of $\mathcal{M}$, is a Lebesgue weak density point of the complement of $\mathcal{M}$.*

The proof of Theorem 1 is based on uniform decay of geometry. Various formulations of this property are known in the literature, when the multiplicity of the critical point is 2, a detailed proof can be found in [7]. It does not generalize to higher degree polynomials, [2], and we do not know if the assertion of Theorem 1 holds for all connectedness loci $\mathcal{M}_d$.

Theorem 1 follows from the following weak version of conformal similarity, see [9]:

**Theorem 2.** *Suppose that $c \in \partial\mathcal{M}$ satisfies the hypothesis of Theorem* 1. *There exists an infinite collection of nested Jordan domains $U \ni c$ and quasi-conformal maps $\Phi_U : U \mapsto \mathbb{C}$, $\Phi_U(c) = c$, with the following properties*:

(1) *the Julia set $\mathcal{J}_c$ has a connected intersection with every closed disk $\overline{U}$,*
(2) *every closed disk $\overline{U}$ contains a continuum $V_U$ such that $V_U \supset \mathcal{J}_c \cap \overline{U}$,*
(3) $\lim_{\operatorname{diam} U\to 0} \frac{1}{\operatorname{diam} U} d_H(V_U, \mathcal{J}_c \cap U) = 0$,
(4) $\lim_{\operatorname{diam} U\to 0} \frac{1}{\operatorname{area}(U)} \operatorname{area}(V_U) = 0$,
(5) *$\Phi_U(V_U) \supset \mathcal{M} \cap \Phi_U(U)$ and $V_U$ is disjoint from the external ray joining infinity and $c$,*
(6) *$\Phi_U$ on $U \setminus V_U$ is equal to $\Psi \circ \Psi_c^{-1}$ where $\Psi_c$ and $\Psi$ are uniforming maps from $\{|z| > 1\}$ of $\hat{\mathbb{C}} \setminus \mathcal{J}_c$ and $\hat{\mathbb{C}} \setminus \mathcal{M}$, respectively, tangent to the identity at $\infty$,*
(7) *the maximal dilation of $\Phi_U$ is bounded independently from $U$ and tends to* 1 *when* $\operatorname{diam} U$ *tends to* 0.

Theorem 2 asserts that there is a system of nested Jordan domains $U \ni c$ such that each $\overline{U} \cap \mathcal{J}_c$ is a connected piece of the Julia set $\mathcal{J}_c$. The continuum $V_U$ can be thougth of as a tiny enlargement of $\overline{U} \cap \mathcal{J}_c$. Every $V_U$ must have a non-empty interior because of the property (5) and the fact that interior of $\mathcal{M}$ is dense in $\mathcal{M}$. The closeness of $V_U$ and $\overline{U} \cap \mathcal{J}_c$ is quantified by (3) and (4). The weak version of conformal similarity between $\mathcal{J}_c$ and $\mathcal{M}$ at $c$ means that there is a system of quasi-conformal maps $\Upsilon_U := \Phi_U^{-1}$ defined on neighborhoods $W_U = \Phi_U(U)$ of $c$ in the parameter space with the property that $\Upsilon_U(\mathcal{M} \cap W_U) \subset V_U$ is close to $J_c \cap U$ both metrically and quasi-conformally, see the properties (3), (4), and (7). The system of similarity maps $\Upsilon_U$ is also consistent in the sense that every $\Upsilon_U$ equals to the same conformal map on the set $W_U \setminus \Phi_U(V_U)$. Because of (7), the similarity becomes asymptotically conformal and means that an asymptotic shape of the Mandelbrot set at $c$ is almost a conformal copy with some 'fuzziness' of the shape of the Julia set $\mathcal{J}_c$ at $c$. The 'fuzziness' cannot be avoided due to topological differences of $\mathcal{M}$ and $\mathcal{J}_c$ at $c$. Namely, the former has always a non-empty interior at $c$ while the latter is a dendrite. A natural question arises whether Theorem 2 can be improved so that the family of maps $(\Phi_U)$ can be replaced by one dynamically canonical and quasi-conformal application. The answer is negative and it follows from the fact that there are parameters $c$ satisfying the hypotheses of Theorem 2 and lying closer and closer to the main cardioid of the small copies of the Mandelbrot set $\mathcal{M}$ at infinitely many different scales nesting at $c$. Therefore, there is infinitely many open disks $D \subset \mathcal{M}$ such that $\lim_{\operatorname{diam} D \to 0} \operatorname{dist}(c, D)/\operatorname{diam} D = 0$, which is an obstruction for a weak similarity in the scale of $D$.

## 3. Harmonic measure and almost conformal similarity

The harmonic measure $\omega$ at $\infty$ of $\partial\mathcal{M}_d$ can be described in terms of the Riemann map

$$\Psi : \hat{\mathbb{C}} \setminus D(0,1) \mapsto \hat{\mathbb{C}} \setminus \mathcal{M}_d$$

which is tangent to identity at $\infty$. Namely, $\Psi$ extends radially almost everywhere on the unit circle with respect to the normalized 1-dimensional Lebesgue measure $\nu$ and $\omega = \Psi_*(\nu)$.

The Hausdorff distance between two sets $E_1$ and $E_2$ in the plane is denoted by $d_H(E_1, E_2)$. Theorem 3 describes $\mathcal{M}_d$ as seen from an average Brownian trajectory in the complement.

**Theorem 3.** *For almost every $c \in \partial\mathcal{M}_d$ with respect to the harmonic measure there exist a Jordan domain $U \ni c$, a continuum $V \ni c$ contained in the closure $\overline{U}$, and a quasi-conformal map $\Phi$ of the plane, $\Phi(c) = c$, with the following properties*:

(1) *$\mathcal{J}_c \cap \overline{U}$ is connected and $\mathcal{J}_c \cap \overline{U} \subset V$,*
(2) $\lim_{r\to 0} \frac{1}{r} d_H(V \cap D(c,r), \mathcal{J}_c \cap D(c,r)) = 0$,
(3) $\lim_{r\to 0} \frac{1}{r^2} \operatorname{area}(V \cap D(c,r)) = 0$,
(4) *$\Phi(V \cap U) \supset \mathcal{M}_d \cap \Phi(U)$ and $V$ is disjoint with the ray from infinity which converges to $c$,*
(5) *$\Phi$ on $U \setminus V$ is equal to $\Psi \circ \Psi_c^{-1}$ where $\Psi_c$ and $\Psi$ are uniforming maps from $\{|z| > 1\}$ of $\hat{\mathbb{C}} \setminus \mathcal{J}_c$ and $\hat{\mathbb{C}} \setminus \mathcal{M}_d$, respectively, tangent to the identity at $\infty$,*
(6) *the maximal dilation of $\Phi$ restricted to $D(c,r)$ tends to* 1 *when $r$ tends to* 0.

Theorem 3 is related to the results of [16] obtained for a certain countable but dense subset of the boundary of the Mandelbrot set (postcritically finite case). A different version of a conformal similarity for Misiurewicz parameters in $\mathcal{M}$ was studied in details in [12].

Theorem 3 asserts that one can glue together the family $(\Phi_U)$ of quasi-conformal homeomorphisms from Theorem 2 to a global similarity map $\Phi$. The continuum $V$ is obtained from the collection of $V_U$ of Theorem 2 by cutting and pasting and can be thougth as a topological enlargement of $\mathcal{J}_c$. As was explained in the previous section, the construction of $\Phi$ is not possible in general due to the topological obstructions. However, we prove that a typical $c$ with respect to harmonic measure lies far away from infinitely renormalizable parameters. This removes the topological obstruction and the construction of $\Phi$ can be carried out.

## 4. Twisting and outside accessibility of $\mathcal{M}_d$

Let $\Omega$ be a domain in $\hat{\mathbb{C}}$ with $\partial\Omega \subset \mathbb{C}$. We say that $w \in \partial\Omega$ is *accessible* from $\Omega$ if there exists a Jordan arc $\gamma \subset \Omega$ terminating at $w$. A point $w$ is *well-accessible* [11] if in addition

$$\operatorname{diam} \gamma(z) \leqslant C \operatorname{dist}(z, \partial\Omega), \tag{2}$$

where $\gamma(z)$ is the subarc of $\gamma$ between $z$ and $w$ ($\operatorname{dist}(z, \partial\Omega)$ is calculated in the spherical metric). We recall that if every point from the boundary of $\Omega$ is well-accessible then $\Omega$ is a John domain. If every point of the boundary of $\Omega$ is well-accessible both from $\Omega$ and $\hat{\mathbb{C}} \setminus \Omega$ then $\Omega$ is a quasidisk.

**Theorem 4.** *For almost every $c \in \partial\mathcal{M}_d$ with respect to the harmonic measure $\omega$, the parameter $c$ is a Lebesgue density point of $\mathbb{C} \setminus \mathcal{M}_d$ but is not well-accessible.*

We will give an outline of the proof of non-accessibility formulated in Theorem 4. Suppose that there exists a set $Z \subset \partial\mathcal{M}_d$ of positive harmonic measure so that every $c \in Z$ is well-accessible. We use Theorem 3 to get that $c \in \mathcal{J}_c$ is also well-accessible for some $c$ from the class of [8]. We amplify this property in a uniform way to the large scale. We obtain that $\mathcal{J}_c$ is uniformly well-accessible at a dense set of points. By passing to the limit, $\mathcal{J}_c$ is the boundary of a John domain and by [3] $c$ is not recurrent, a contradiction.

Another application of Theorem 4 is inspired by the McMillan twist theorem (Theorem 6.18 in [11]). We say that $c \in \partial\mathcal{M}_d$ is *twisting* if for every curve $\gamma$ landing at $c$

$$\liminf_{z\to c,\ z\in\gamma} \arg(z-c) = -\infty \quad \text{and} \quad \limsup_{z\to c,\ z\in\gamma} \arg(z-c) = \infty.$$

**Corollary 4.1.** *For almost every $c \in \partial\mathcal{M}_d$ with respect to the harmonic measure $\omega$, the parameter $c$ is twisting.*

**Proof.** The McMillan twist theorem implies that almost every point of $c \in \partial\mathcal{M}_d$ is either well-accessible or twisting. By Theorem 4, the set of $c \in \mathcal{M}_d$ which are well-accessible is of harmonic measure zero. □

## 5. Lyapunov exponents

For every $c \in \partial\mathcal{M}_d$, let $\omega_c$ be the harmonic measure at $\infty$ of $\mathcal{J}_c$. The measure $\omega_c$ is known to be a unique measure of maximal entropy for $f_c$ and its Lyapunov exponent $\int \log|f_c'|\,\mathrm{d}\omega_c$ is equal to $\log d$. A point $z \in \mathcal{J}_c$ is typical with respect to the harmonic measure $\omega_c$ if for every continuous function $H$ in $\mathcal{J}_c$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} H\bigl(f_c^i(z)\bigr) = \int H\,\mathrm{d}\omega_c.$$

**Theorem 5.** *For almost every $c \in \partial\mathcal{M}_d$ with respect to the harmonic measure $\omega$, the critical value $f_c(c)$ is typical for the corresponding harmonic measure $\omega_c$ on $\mathcal{J}_c$ and its Lyapunov exponent*

$$\lim_{n\to\infty} \frac{1}{n}\bigl|\log\bigl(f_c^n\bigr)'(c)\bigr|$$

*exists and is equal to* $\log d$.

**Proof.** A typical $c$ in $\partial\mathcal{M}_d$ with respect to $\omega$ has an external angle $\gamma$ such that the orbit of $\gamma$ by $F = z^d$ is typical for the normalized 1-dimensional Lebesgue measure $\nu$ in the unit circle. By [8,15],

$$\liminf_{n\to\infty} \frac{1}{n}\log\bigl|\bigl(f_c^n\bigr)'(c)\bigr| > 0$$

and [6] implies that the Riemann map $\Psi_c$ onto $\hat{\mathbb{C}} \setminus \mathcal{J}_c$ is Hölder continuous. In particular, it has a continuous extension to the unit circle. Let $H$ be a continuous function defined in $\mathcal{J}_c$. Then $h := H \circ \Psi_c$ is a continuous function in the unit circle. Using that $h(F^i(\gamma)) = H \circ f_c^i(c)$ and $\omega_c = (\Psi_c)_*(\nu)$, we obtain that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} H\bigl(f_c^i(c)\bigr) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} h\bigl(F^i(\gamma)\bigr) = \int h\,\mathrm{d}\nu = \int H\,\mathrm{d}\omega_c.$$

The function $\log|f_c'|$ is continuous everywhere in $\mathcal{J}_c$ except at 0 where it is equal to $-\infty$. Still it is integrable and by the Volume Lemma, $\int_{\mathcal{J}_c} \log|f_c'|\,\mathrm{d}\omega_c = \log d$. Theorem 5 follows from the fact that $f_c(c)$ is typical with respect to $\omega_c$ [8]. □

Theorem 5 has an analogue in 1-dimensional real dynamics, see [1]. A probabilistic explanation of Theorem 5 is that a Brownian particle in $\hat{\mathbb{C}} \setminus \mathcal{M}_d$ lands typically at $c \in \partial\mathcal{M}_d$ such that $f_c^i(c)$ for large $i$ behaves as a random variable with distribution given by the corresponding harmonic measure $\omega_c$. This property is dynamically quantifiable in terms of distortion estimates, see [8], and leads to a direct proof of Theorem 5, i.e. without invoking the continuity results of [6].

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