C. R. Acad. Sci. Paris, Ser. I 349 (2011) 119–122

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Combinatorics/Algebra

A short proof of Kontsevich's cluster conjecture ☆

Une courte démonstration d'une conjoncture de Kontsevich

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ARTICLE INFO

Article history:
Received 18 November 2010
Accepted after revision 4 January 2011
Available online 20 January 2011

Presented by Maxime Kontsevich

ABSTRACT

We give an elementary proof of the Kontsevich conjecture that asserts that the iterations of the noncommutative rational map $K_r : (x, y) \mapsto (xyx^{-1}, (1 + y^r)x^{-1})$ are given by noncommutative Laurent polynomials.

RÉSUMÉ

Nous proposons une démonstration élémentaire d'une conjoncture de Kontsevich qui affirme que l'itération de l'application non-commutative rationnelle $K_r : (x, y) \mapsto (xyx^{-1}, (1+y^r)x^{-1})$ est donnée par des polynômes de Laurent non-commutatifs.

The aim of this Note is to give an elementary proof of the following Kontsevich conjecture.

Recall that the *Kontsevich map* K_r, $r \in \mathbb{Z}_{>0}$, is the following (birational) automorphism of a noncommutative plane (i.e., of the free skew-field with generators x and y):

$$K_r : (x, y) \mapsto \bigl(xyx^{-1}, \bigl(1 + y^r\bigr)x^{-1}\bigr).$$

Conjecture 1 *(M. Kontsevich). For any $r_1, r_2 \in \mathbb{Z}_{>0}$ all iterations $\underbrace{\cdots K_{r_1} K_{r_2} K_{r_1}}_{k}(x, y)$, $k \geqslant 1$, are given by noncommutative Laurent polynomials in x and y.*

The conjecture extends the rank 2 Laurent phenomenon established previously for cluster algebras and their quantum versions (see [4,6,5,2,1]) to the "fully noncommutative" setting.

The Kontsevich conjecture was first proved for $r_1 = r_2 = 2$ by A. Usnich in [8] and was later settled by A. Usnich in [9] in greater generality when $r_1 = r_2 = r$ (with $1 + y^r$ replaced by any monic palindromic polynomial $H(y)$) by means of derived categories. Independently, Conjecture 1 was verified for $(r_1, r_2) \in \{(2, 2), (4, 1), (1, 4)\}$ in [3] along with the positivity conjecture: for $(r_1, r_2) \in \{(2, 2), (4, 1), (1, 4)\}$ all noncommutative Laurent polynomials in question have nonnegative integer coefficients.

Our goal is to give a short proof of Conjecture 1 by adjusting the ideas of [4,6,5,2,1] to the noncommutative rank 2 setting. Note, however, that there is currently no theory of "fully noncommutative" cluster algebras, as well as a higher rank

☆ The authors were supported in part by the NSF grant DMS #0800247.
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1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.01.004

Kontsevich conjecture. We are planning to construct examples of such higher rank noncommutative clusters elsewhere and thus provide some higher rank analogues of the Kontsevich conjecture.

To present our proof of Conjecture 1, we need some notation. Denote

$$(x_k, y_k) := \underbrace{\cdots K_{r_1} K_{r_2} K_{r_1}}_{k}(x, y)$$

and denote $z := [x, y] = xyx^{-1}y^{-1}$. Then it is easy to see by induction that $[x_k, y_k] = [x, y] = z$ for all k. This taken together with the recursion $x_{k+1} = x_k y_k x_k^{-1}$ and $y_{k+1} = (1 + y_k^{r_k})x_k^{-1}$, where

$$r_k = \begin{cases} r_1 & \text{if } k \text{ is odd,} \\ r_2 & \text{if } k \text{ is even,} \end{cases} \tag{1}$$

gives the following three recursions (they first appeared in [3, Section 2.2])

$$x_{k+1} = zy_k,$$

$$y_{k+1} z y_{k-1} = 1 + y_k^{r_k}, \tag{2}$$

$$y_{k+1} z y_k = y_k y_{k+1}. \tag{3}$$

Let $\mathcal{F}_2 = \mathbb{Q}\langle y_1^{\pm 1}, y_2^{\pm 1}\rangle$ be the group algebra of the free group in 2 generators. It was proved by A.I. Malcev (see, e.g., [7, Section 8.7]) that $\mathcal{F}_2$ is a *divisible algebra*, i.e., it embeds in a division ring (we denote the smallest one by $Frac(\mathcal{F}_2)$).

Define elements $y_k \in Frac(\mathcal{F}_2)$, $k \in \mathbb{Z} \setminus \{1, 2\}$, recursively by Eq. (2) where $z := [y_2^{-1}, y_1] = y_2^{-1} y_1 y_2 y_1^{-1}$.

Note that $y_0, y_3 \in \mathcal{F}$ and let $\mathcal{A} = \mathcal{A}(r_1, r_2)$ be the subalgebra of $\mathcal{F}$ generated by $y_0, y_1, y_2, y_3, z, z^{-1}$. We will refer to $\mathcal{A}$ as a (*purely*) *noncommutative cluster algebra* of type (r_1, r_2).

Lemma 2. *The elements $y_k \in Frac(\mathcal{F}_2)$ satisfy for all $k \in \mathbb{Z}$ Eq.* (3).

Proof. Indeed, (3) is obvious for $k = 1$. Let us prove it for $k \geqslant 1$ by induction. We will use the inductive hypothesis in the form $y_k y_{k-1}^{-1} z^{-1} = y_{k-1}^{-1} y_k$. Indeed, since $y_{k+1} z = (1 + y_k)^{r_k} y_{k-1}^{-1}$, we obtain

$$\begin{aligned} y_{k+1} z y_k - y_k y_{k+1} &= (1 + y_k)^{r_k} y_{k-1}^{-1} y_k - y_k (1 + y_k)^{r_k} y_{k-1}^{-1} z^{-1} \\ &= (1 + y_k)^{r_k} y_{k-1}^{-1} y_k - (1 + y_k)^{r_k} y_k y_{k-1}^{-1} z^{-1} \\ &= (1 + y_k)^{r_k} y_{k-1}^{-1} y_k - (1 + y_k)^{r_k} y_{k-1}^{-1} y_k = 0 \end{aligned}$$

by the inductive hypothesis. The relation (3) for $k \leqslant 0$ also follows. $\square$

Thus, based on the above discussion, Conjecture 1 directly follows from our main result.

Main Theorem 3. *Each y_k belongs to $\mathcal{A}$, e.g., y_k is a noncommutative Laurent polynomial in y_1, y_2.*

Proof. Denote by $\mathcal{A}_k = \mathcal{A}_k(r_1, r_2)$ the subalgebra of $\mathcal{F}_2$ generated by $y_k, y_{k+1}, y_{k+2}, y_{k+3}, z^{\pm 1}$. It suffices to prove the following result (which is a noncommutative version of [2, Formula (4.12)] and [1, Lemma 5.8]).

Theorem 4. *$\mathcal{A}_k = \mathcal{A}$ for all $k \in \mathbb{Z}$.*

Proof. Since $\mathcal{A} = \mathcal{A}_0$, it suffices to prove that $\mathcal{A}_k = \mathcal{A}_{k+1}$ for $k \in \mathbb{Z}$, i.e., that for all $k \in \mathbb{Z}$ one has

$$y_{k+4} \in \mathcal{A}_k, \qquad y_k \in \mathcal{A}_{k+1}. \tag{4}$$

Proposition 5. *For each $n \in \mathbb{Z}$ one has: $y_{k+4} z = z y_k (y_{k+3} z)^{r_{k+1}} - \sum_{j=0}^{r_{k+1}-1} (z y_{k+1})^j z (y_{k+2} z)^{r_k - 1} (y_{k+3} z)^j$.*

Proof. For simplicity (and without loss of generality) we assume that $k = 0$. We start with the following technical result.

Lemma 6. *For each $m \geqslant 0$ we have: $y_1^m (y_3 z)^m = 1 + \sum_{k=0}^{m-1} y_1^k (y_2 z)^{r_2} (y_3 z)^k$.*

Proof. We proceed by induction on m. For $m = 0$ the assertion is clear. Assume that $m > 0$ and it holds for $m - 1$. Let us prove it for m. Note that (2) and (3) imply that

$$y_{k-1} y_{k+1} z = 1 + (y_k z)^{r_k}. \tag{5}$$

Indeed, using (5), we obtain

$$y_1^m(y_3z)^m = y_1^{m-1}(y_1y_3z)(y_3z)^{m-1} = y_1^{m-1}\big(1+(y_2z)^{r_2}\big)(y_3z)^{m-1} = y_1^{m-1}(y_2z)^{r_2}(y_3z)^{m-1} + y_1^{m-1}(y_3z)^{m-1}$$
$$= y_1^{m-1}(y_2z)^{r_2}(y_3z)^{m-1} + 1 + \sum_{k=0}^{m-2} y_1^k(y_2z)^{r_2}(y_3z)^k = 1 + \sum_{k=0}^{m-1} y_1^k(y_2z)^{r_2}(y_3z)^k .$$

The lemma is proved. □

Furthermore, compute:

$$y_4z = y_2^{-1}\big((y_3z)^{r_1}+1\big) = y_2^{-1}(y_3z)^{r_1} + y_2^{-1} = \big(zy_0 - y_2^{-1}(y_1)^{r_1}\big)(y_3z)^{r_1} + y_2^{-1}$$
$$= zy_0(y_3z)^{r_1} - y_2^{-1}\big(y_1^{r_1-1}(y_1y_3z)(y_3z)^{r_1-1} - 1\big) = zy_0(y_3z)^{r_1} - y_2^{-1}\big(y_1^{r_1-1}\big(1+(y_2z)^{r_2}\big)(y_3z)^{r_1-1} - 1\big).$$

We have:

$$y_1^{r_1-1}\big(1+(y_2z)^{r_2}\big)(y_3z)^{r_1-1} - 1 = y_1^{r_1-1}(y_2z)^{r_2}(y_3z)^{r_1-1} + y_1^{r_1-1}(y_3z)^{r_1-1} - 1.$$

Using Lemma 6 and taking into account that $y_1^m y_2 = y_2(zy_1)^m$ for $m > 0$, we obtain:

$$y_1^{r_1-1}\big(1+(y_2z)^{r_2}\big)(y_3z)^{r_1-1} - 1 = y_1^{r_1-1}(y_2z)^{r_2}(y_3z)^{r_1-1} + \sum_{k=0}^{r_1-2} y_1^k(y_2z)^{r_2}(y_3z)^k = \sum_{k=0}^{r_1-1} y_1^k(y_2z)^{r_2}(y_3z)^k$$
$$= y_2\sum_{k=0}^{r_1-1}(zy_1)^k z(y_2z)^{r_2-1}(y_3z)^k.$$

Therefore, $y_4z = zy_0(y_3z)^{r_1} - \sum_{k=0}^{r_1-1}(zy_1)^k z(y_2z)^{r_2-1}(y_3z)^k$. This proves Proposition 5. □

Proposition 5 gives us the first inclusion (4). Prove second inclusion (4) now. We need the following obvious fact. Let σ be the anti-automorphism of $\mathcal{F}_2$ given by: $\sigma(y_1) = y_2$, $\sigma(y_1) = y_2$ (so that $\sigma(z) = z$).

Lemma 7. *$\sigma(y_k) = y_{3-k}$ for $k \in \mathbb{Z}$, in particular, $\sigma(\mathcal{A}_k(r_1, r_2)) = \mathcal{A}_{-k}(r_2, r_1)$ for $k \in \mathbb{Z}$.*

This immediately implies the second inclusion (4): $y_{1-k} \in \mathcal{A}_{-k}$, $k \in \mathbb{Z}$, and Theorem 4 is proved. □

Therefore, Theorem 3 is proved. □

And, ultimately, Conjecture 1 is proved.

Example 8. Let $r_1 = r_2 = 2$. We have: $y_{k+1}zy_{k-1} = y_k^2 + 1$, $y_{k-1}y_{k+1}z = y_kzy_kz + 1$ for all $k \in \mathbb{Z}$. This implies:

$$y_4z = y_2^{-1}(y_3zy_3z + 1) = \big(zy_0 - y_2^{-1}y_1^2\big)y_3(zy_3z) + y_2^{-1}$$
$$= zy_0y_3zy_3z - y_2^{-1}\big(y_1(y_1y_3z)y_3z - 1\big).$$

Note that $y_1(y_1y_3z)y_3z - 1 = y_1(y_2zy_2z + 1)y_3z - 1 = y_1y_2zy_2zy_3z + y_1y_3z - 1 = y_2zy_1zy_2zy_3z + (y_2z)^2$. Therefore,

$$y_4z = zy_0(y_3z)^2 - (zy_1zy_2zy_3z + zy_2z).$$

The noncommutative cluster algebra $\mathcal{A} = \mathcal{A}(r_1, r_2)$ has a number symmetries in addition to the anti-involution $\sigma : \mathcal{A}(r_1, r_2) \widetilde{\to} \mathcal{A}(r_2, r_1)$ from Lemma 7: the translation $y_k \mapsto y_{k+1}$, $k \in \mathbb{Z}$, defines an isomorphism $\tau : \mathcal{A}(r_1, r_2) \widetilde{\to} \mathcal{A}(r_2, r_1)$, which is an automorphism when $r_1 = r_2$.

We conclude with a brief discussion of the presentation of $\mathcal{A}$.

Proposition 9. *The generators $y_0, y_1, y_2, y_3, z^{\pm 1}$ of $\mathcal{A}$ satisfy (for $i = 0, 1, 2$, $j = 1, 2$):*

$$y_iy_{i+1} = y_{i+1}zy_i, \qquad y_{j+1}zy_{j-1} = y_j^{r_j} + 1,$$
$$y_{j-1}y_{j+1}z = (y_jz)^{r_j} + 1, \qquad y_3zy_0 - zy_0y_3z = y_2^{r_2-1}y_1^{r_1-1} - z(y_1z)^{r_1-1}(y_2z)^{r_2-1}.$$

Proof. Only the last relation needs to be proved (the first three relations are (3), (2), and (5) respectively). Indeed, using the available relations in $\mathcal{F}_2$, we obtain:

$$\begin{aligned} y_0 y_3 z &= \big(\big(1+(y_1 z)^{r_1}\big)z^{-1}y_2^{-1}\big)\big(y_1^{-1}\big(1+(y_2 z)^{r_2}\big)\big) \\ &= \big(1+(y_1 z)^{r_1}\big)z^{-1}y_1^{-1}z^{-1}y_2^{-1}\big(1+(y_2 z)^{r_2}\big) = h_{r_1}(y_1 z)h_{r_2}(y_2 z), \end{aligned}$$

where $h_r(y) = y^{-1} + y^{r-1}$. Similarly,

$$y_3 z y_0 = \big(\big(1+y_2^{r_2}\big)y_1^{-1}\big)\big(z^{-1}y_2^{-1}\big(1+y_1^{r_1}\big)\big) = \big(1+y_2^{r_2}\big)y_2^{-1}y_1^{-1}\big(1+y_1^{r_1}\big) = h_{r_2}(y_2)h_{r_1}(y_1).$$

Taking into account that $y_1 y_2 y_1^{-1} = y_2 z$ and $y_2^{-1} y_1 y_2 = z y_1$, we obtain:

$$\begin{aligned} y_3 z y_0 &= y_2^{r_2-1}y_1^{r_1-1} + h_{r_2}(y_2)y_1^{-1} + y_2^{-1}y_1^{r_1-1} + y_2^{-1}y_1^{-1} \\ &= y_2^{r_2-1}y_1^{r_1-1} + (zy_1)^{r_1-1}y_2^{-1} + y_1^{-1}h_{r_2}(y_2 z) + y_1^{-1}z^{-1}y_2^{-1} \\ &= y_2^{r_2-1}y_1^{r_1-1} + z(y_1 z)^{r_1-1}(y_2 z)^{-1} + z(y_1 z)^{-1}h_{r_2}(y_2 z) + z(y_1 z)^{-1}(y_2 z)^{-1} \\ &= y_2^{r_2-1}y_1^{r_1-1} + z(y_1 z)^{r_1-1}(y_2 z)^{r_2-1}. \end{aligned}$$

The proposition is proved. □

We expect that the relations in Proposition 9 are defining.

Acknowledgements

This work started when the authors were visiting IHES in July 2010. We thank Maxim Kontsevich for his kind hospitality and stimulating discussions.

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