A class of existence results for the singular Liouville equation

Une classe de résultats d’existence pour l’équation de Liouville singulière

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Abstract

We consider a class of elliptic PDEs on closed surfaces with exponential nonlinearities and Dirac deltas on the right-hand side. The study arises from abelian Chern–Simons theory in self-dual regime, or from the problem of prescribing the Gaussian curvature in presence of conical singularities. A general existence result is proved using global variational methods: the analytic problem is reduced to a topological problem concerning the contractibility of a model space, the so-called space of formal barycenters, characterizing the very low sublevels of a suitable functional.

Résumé

Nous considérons une classe d’EDP elliptiques sur une surface compacte et sans bord, avec une nonlinéarité exponentielle et des masses de Dirac dans le membre de droite. Ce travail est motivé par l’étude d’équations de Chern–Simons abéliennes en régime auto-dual, ainsi que par le problème de la courbure gaussienne prescrite pour des surfaces avec singularités coniques. Nous démontrons un résultat général d’existence en utilisant des méthodes variationnelles globales : le problème analytique est réduit à un problème topologique concernant la contractibilité d’un espace modèle, l’espace des barycentres formels, qui caractérise les sous-niveaux très bas d’une fonctionnelle appropriée.

Version française abrégée

Dans ce travail nous étudions l’équation

\[-\Delta g u = \rho \left( \frac{h(x)e^{2u}}{\int_{\Sigma} h(x)e^{2u} \, dV_g} - 1 \right) - 2\pi \sum_{i=1}^{m} \alpha_j (\delta_{p_j} - 1)\]

sur une surface compacte \((\Sigma, g)\). \(\rho\) est un paramètre positif et \(h : \Sigma \to \mathbb{R}\) est une fonction positive et régulière. Les \(p_j\) sont des points dans \(\Sigma\) et les \(\alpha_j\) sont des nombres réels. L’étude de l’équation précédente est motivée par la théorie des équations de Chern–Simons abéliennes en régime auto-dual ainsi que par le problème de la courbure gaussienne prescrite pour des surfaces avec singularités coniques. En utilisant des inégalités du type Moser–Trudinger et des fonctions-test appropriées, le problème est réduit à l’étude de la topologie d’un ensemble approprié. Cet ensemble coïncide avec les barycentres formels.
1. Introduction

This article deals with the study of a class of equations having the form

\[-\Delta_g u = \rho \left( \frac{h(x)e^{2u}}{\int_{\Sigma} h(x)e^{2u} \, dV_g} - 1 \right) - 2\pi \sum_{j=1}^{m} \alpha_j (\delta_{p_j} - 1)\]

(1)
on a Riemannian 2-manifold \((\Sigma, g)\). Here \(\rho\) is a positive parameter, \(h : \Sigma \to \mathbb{R}\) a smooth positive function, the \(\alpha_j\)'s are real numbers, \(p_j \in \Sigma\) are some fixed points and the subscripts \((\cdot)_g\) refer to the metric \(g\).

This equation arises in different contexts and has therefore been object of intense study in the last decades. First of all, it naturally appears in the study of multivortices in the Electroweak Theory by Glashow–Salam–Weinberg [18], where \(u\) can be interpreted as the logarithm of the absolute value of the wave function and the points \(p_j\)'s are the vortices, where the wave function vanishes. Secondly, this class of equations has proved to be relevant in many other physical frames, such as the study of the statistical mechanics of point vortices in the mean field limit [17,6,7] and the abelian Chern–Simons Theory, as discussed in [15,25]. Eq. (1) also admits a geometric interpretation, which is however much more evident if we restrict our attention to the regular case

\[-\Delta_g u = \rho \left( \frac{h(x)e^{2u}}{\int_{\Sigma} h(x)e^{2u} \, dV_g} - 1 \right).\]

(2)

Indeed, if we consider a conformal change of metric, say \(\hat{g} = e^{2w}g\), the Gaussian curvature transforms according to the law

\[K_{\hat{g}} = e^{-2w}(-\Delta_g w + K_g),\]

(3)

and so, if \(K_g\) is constant, the problem of asking whether \(g\) can be conformal to a metric with Gaussian curvature a given function \(K_{\hat{g}}\) is equivalent to (2). This is the famous Kazdan–Warner problem [16], also known as Nirenberg problem in the special case when \((\Sigma, g)\) is just the standard sphere. While moving from Problem (2) to Problem (1), the extra terms can be viewed as singularities in the Gauss curvature corresponding to a local conical structure, as can be justified via an extension of the Gauss–Bonnet formula (see [26]). With a non-constant function \(h(x)\), we are considering the problem of prescribing the Gaussian curvature through conformal metrics. This is a question dual to the uniformization problem, where on general surfaces one looks for metrics with constant Gaussian curvature (in higher dimension this is the Yamabe problem, see the survey [19]).

Mainly due to its natural geometric appeal, Eq. (2) has been widely studied in the last forty years. Because of its strong nonlinearity, it has been tackled by means of two classes of techniques both leading to quite sophisticated results: on the one hand, topological methods relying on the degree theory by Leray–Schauder (see [9,10,20], and also [11] for the singular case), on the other purely variational methods based on an improvement of the Moser–Trudinger inequality. Indeed (2) is the Euler–Lagrange equation associated to the \(C^1\) functional

\[J_\rho(u) = \int_{\Sigma} |\nabla_g u|^2 \, dV_g + 2\rho \int_{\Sigma} J_g u \, dV_g - \rho \log \int_{\Sigma} h(x)e^{2u} \, dV_g,\]

defined on the Sobolev space \(H^1(\Sigma, g)\). The weak form of the Moser–Trudinger inequality

\[\log \int_{\Sigma} e^{2(u - \alpha)} \, dV_g \leq \frac{1}{4\pi} \int_{\Sigma} |\nabla_g u|^2 \, dV_g + C_{\Sigma, g} \quad u \in H^1(\Sigma, g)\]

(4)
guarantees that \(J_\rho\) is well-defined on \(H^1(\Sigma, g)\) for any value of \(\rho \in \mathbb{R}\). Moreover, \(J_\rho\) is lower semi-continuous with respect to the weak topology of that space and so, since (4) gives coercivity of \(J_\rho\) if \(\rho < 4\pi\), we immediately get existence of critical points for this range of values and the corresponding solvability of (2). It is clear that such critical points are global minima for \(J_\rho\). Such a direct variational approach does not apply to the case \(\rho \geq 4\pi\) as can be seen by exhibiting explicit examples (i.e. standard bubbling functions). This is the reason why more sophisticated tools are needed in this regime. Such a variational structure can be recovered for the singular problem by performing a change of variables like

\[u \rightarrow u + \sum_{j=1}^{m} \alpha_j \log d_\hat{g}(x, p_j)\]

(5)

so that our singular equation can be rewritten as.
Proposition 1. Assume \( \rho \in (4k\pi, (4k+1)\pi) \) for some \( k \geq 1 \). Then, for any \( \epsilon > 0 \) and \( r > 0 \) there exists a sufficiently large positive constant \( L := L(\epsilon, r) \) such that for every \( u \in H^1(\Sigma, g) \) with \( J_\rho(u) \leq -L \) there are \( k \) points on \( \Sigma \) (say \( p_{1,u}, \ldots, p_{k,u} \)) so that

\[
\frac{\int_{\Sigma \setminus \bigcup_{i=1}^{k} B_{\delta}(p_{i,u})} e^{2u} dV_g}{\int_{\Sigma} e^{2u} dV_g} < \epsilon.
\]

By some hard work, this leads to the definition of a projection operator from the very low sublevels of \( J_\rho \) (say \( J_\rho^{-L}, L \gg 1 \)) to the space of formal barycenters

\[
\Sigma_k := \left\{ \sum_{i=1}^{k} t_i \delta_{p_i} : \sum_{i=1}^{k} t_i = 1, \ t_i \geq 0, \ p_i \in \Sigma \ \forall i \in \{1, \ldots, k\} \right\}
\]
and indeed this is proved to be a homotopy equivalence (see [14] and [22]). As a consequence, the non-contractibility of
\( J_{\rho}^{-1} \) (for \( L \) big enough) is proved by studying the homology of the model space \( \Sigma_k \).

The first step of our study was then a similar improved inequality that is based on both (4) and (8) and is proved by
means of cut-off functions and a suitable spectral decomposition.

**Lemma 2.** Let \( n \in \mathbb{N} \) and let \( I \subseteq \{1, \ldots, m\} \) with \( n + \text{card}(I) > 0 \), where \( \text{card}(I) \) denotes the cardinality of a set. Assume there exists
\( r > 0 \), \( \delta_0 > 0 \) and pairwise distinct points \( \{q_1, \ldots, q_n\} \subseteq \Sigma \setminus \{p_1, \ldots, p_m\} \) such that:

- For any couple \((a, b) \subseteq \{q_1, \ldots, q_n\} \cup \left( \bigcup_{i \in I} p_i \right) \) with \( a \neq b \) one has \( \text{dist}_g(B_r(a), B_r(b)) \geq \delta_0 \);
- For any \( a \in \{q_1, \ldots, q_n\} \) one has \( \text{dist}_g(p_i, B_r(a)) \geq \delta_0 \) for any \( i \in \{1, \ldots, m\} \setminus I \);

and consider any \( \gamma_0 \in (0, \frac{1}{\pi \cdot \text{card}(I)}) \).

Then, for any \( \tilde{\epsilon} > 0 \) there exists a constant \( C := C(\Sigma, g, n, I, r, \delta_0, \gamma_0, \tilde{\epsilon}) \) such that
\[
\log \int_\Sigma \tilde{h} e^{(u-\tilde{\epsilon})} \, dV_g \leq \frac{1}{4 \pi (n + \sum_{i \in I} (1 + \alpha_i) - \tilde{\epsilon})} \int_\Sigma |\nabla_g u|^2 \, dV_g + C,
\]
for all functions \( u \in H^1(\Sigma) \) satisfying
\[
\frac{\int_{B_r(a)} \tilde{h} e^{2u} \, dV_g}{\int_\Sigma \tilde{h} e^{2u} \, dV_g} \geq \gamma_0, \quad \forall a \in \left\{ q_1, \ldots, q_n \cup \left( \bigcup_{i \in I} p_i \right) \right\}.
\]

Following the guide of the regular case, we are then led to claim the structure of the very low sublevels of the functional
\( J_{\rho, g} \).

**Definition 1.** Given a point \( q \in \Sigma \) we define its weighted cardinality as follows:
\[
\chi(q) = \begin{cases} 1 + \alpha_j & \text{if } q = p_j \text{ for some } j = 1, \ldots, m; \\ 1 & \text{otherwise.} \end{cases}
\]

The cardinality of any finite set of (pairwise distinct) points on \( \Sigma \) is obtained extending \( \chi \) by additivity.

This enables us to easily describe selection rules to determine admissibility conditions for specific barycentric configurations
in dependence on the values of the \( \alpha_j \)'s and \( \rho \).

**Definition 2.** Suppose all the parameters \( \rho, \alpha_1, \ldots, \alpha_m \) are fixed. We define the corresponding space of formal barycenters as follows
\[
\Sigma_{\rho, g} = \left\{ \sum_{j \in J} t_j \delta_{q_j} : \sum_{j \in J} t_j = 1, \ t_j \geq 0, \ q_j \in \Sigma \, \pi(\chi(j)) < \rho \right\}.
\]

We expect that \( \Sigma_{\rho, g} \) is homotopy equivalent to the very low sublevels of the functional \( J_{\rho, g} \); we are indeed able to
define a non-trivial projection operator \( \Pi : J_{\rho, g}^{-1} \to \Sigma_{\rho, g} \) (for some appropriate choice of \( L \)) and an immersion \( \Phi : \Sigma_{\rho, g} \to J_{\rho, g}^{-1} \phi \) so that the composition \( \Pi \circ \Phi : \Sigma_{\rho, g} \to J_{\rho, g}^{-1} \phi \) is (homotopy) equivalent to the identity on the same space. Although this
fact does not imply the homotopic equivalence, it is however sufficient for our purposes. For the sake of clarity, assume
from now onwards that the parameter \( L \) is fixed so that \( \Pi \) may be defined and the gauge of \( \Phi \) (which is, to be precise,
a one-parameter family of immersion operators \( \Phi_\lambda, \lambda \in \mathbb{R}_{\geq 0} \)) is fixed so that \( \Phi = \Phi_\lambda \) takes values in \( J_{\rho, g}^{-2\lambda} \). Moreover, let us
write \( \varphi_{\alpha} \) instead of \( \Phi(\sigma) \) for \( \sigma \in \Sigma_{\rho, g} \).

**Remark 1.**
- The definition of the immersion \( \Phi \) was first done in [23] and, naively, associates to a given measure of \( \Sigma_{\rho, g} \) a corresponding
generalized multipole bubbling functions. This is a delicate point because some sort of smart interpolation
between the regular and the singular bubbling functions is needed.
- In order to define such \( \Pi \) we follow the line of [14] but need to deal with the very sophisticated topological structure
of \( \Sigma_{\rho, g} \) as a stratified set.

These are the tools we need to prove our main result, which is essentially an existence theorem for non-critical values
of \( \rho \) (depending on \( g \)), see the denominators in (13).
Definition 3. We say that \( \overline{\rho} \in \mathbb{R}_{>0} \) is a singular value for Problem (1) if it is representable as follows:

\[
\overline{\rho} = 4\pi n + 4\pi \sum_{i \in I} (1 + \alpha_i)
\]

for some \( n \in \mathbb{N} \) and \( I \subseteq \{1, \ldots, m\} \) (possibly empty) satisfying \( n + \text{card}(I) > 0 \). The set of singular values will be denoted by \( \overline{\Sigma} = \overline{\Sigma}(\alpha) \).

Theorem 1. Suppose that the parameters \( \alpha \in (-1, 0)^m \) and \( \rho \in \mathbb{R}_{>0} \setminus \overline{\Sigma} \) are such that the set \( \Sigma_{\rho, \alpha} \) is not contractible with respect to the topology of \( C^1(\Sigma, g)^n \). Then Problem (1) admits a solution \( u \in H^1(\Sigma, g) \). Moreover, we have that \( u = v + \sum_{j=1}^m \alpha_j G_{\rho_j} \) with \( G_{\rho_j} \) the Green functions defined above and \( v \in C^\nu(\Sigma, g) \) for any \( \nu \in [0, \gamma_0) \) with \( \gamma_0 \in (0, 1) \) solving Eq. (6).

3. Outline of the proof

Our plan is to use a general min–max scheme in the form of a suitable topological cone construction.

1. Minimax scheme. We define the topological cone over \( \Sigma_{\rho, \alpha} \) as follows:

\[
\Theta_{\rho, \alpha} = (\Sigma_{\rho, \alpha} \times [0, 1]) / (\Sigma_{\rho, \alpha} \times \{1\}),
\]

where we are identifying all the points in \( \Sigma_{\rho, \alpha} \times \{1\} \). Consequently, we consider the family of continuous maps

\[
\mathcal{H}_{\rho, \alpha} = \{ h : \Theta_{\rho, \alpha} \to H^1(\Sigma, g) : h(\sigma) = \varphi_\sigma \text{ for every } \sigma \in \Sigma_{\rho, \alpha} \},
\]

and then the number

\[
\overline{H}_{\rho, \alpha} = \inf_{h \in \mathcal{H}_{\rho, \alpha}} \sup_{\sigma \in \Theta_{\rho, \alpha}} J_{\rho, \alpha}(h(\sigma)).
\]

We claim that under the assumption of Theorem 1 one has \( \overline{H}_{\rho, \alpha} \geq -L \). It is worth proving first that the class \( \mathcal{H}_{\rho, \alpha} \) is not empty. To this aim, notice that the map \( h(\sigma, t) = (1 - t)\varphi_\sigma \), \( (\sigma, t) \in \Sigma_{\rho, \alpha} \), belongs to \( \mathcal{H}_{\rho, \alpha} \).

Concerning the lower bound on the min–max value, we just need to argue by contradiction. If \( \overline{H}_{\rho, \alpha} < -L \), then there should be a map \( h \) such that its image \( h(\Theta_{\rho, \alpha}) \) (which is a topological cone in \( H^1(\Sigma, g) \)) would be in \( J_{\rho, \alpha}^{-L} \). As a consequence, the composite map

\[
t \to \Pi(h(\sigma, t)), \quad \sigma \in \Sigma_{\rho, \alpha}
\]

would be a homotopy equivalence between \( \Pi(\overline{h}(0, \sigma)) = \Pi \cdot \Phi(\sigma) \) and a constant map. On the other hand, we know that the function \( \Pi \cdot \Phi(\sigma) \) is homotopic to the identity in \( \Sigma_{\rho, \alpha} \) and hence, by composition the space \( \Sigma_{\rho, \alpha} \) would be contractible, a contradiction. Hence we deduce \( \overline{H}_{\rho, \alpha} \geq -L \).

2. Existence on a dense set. We then show that the scheme outlined in the previous step leads to existence for a dense set of \( \rho \)'s (in a suitable neighborhood of a fixed value). This relies on a monotonicity trick by Struwe [24] and exploited also in [13].

3. Conclusion via blow-up analysis. To conclude, we need to build a sequence of solutions for approximating values of \( \rho \) and extract a converging sub-sequence via blow-up analysis. This may be done thanks to the results in [4], generalized in [3] specifically for the case of negative parameters, and extending previous estimates in [5,21,20].

4. Final remark

Theorem 1 reduces an analytic problem (existence for (1)) to a topological problem (contractibility of \( \Sigma_{\rho, \alpha} \)). As a result, we would hope such a space to be non-contractible for generic values of the parameters \( \rho \) and \( \alpha \). Indeed, this seems to be true. More precisely, we conjecture that the cases of contractibility of \( \Sigma_{\rho, \alpha} \) can be classified so that, very surprisingly, such topological question may be reduced to test a simple algebraic relation involving \( \rho \) and \( \alpha \). An outline of our conjecture would first require a description of the stratifies structure of \( \Sigma_{\rho, \alpha} \) and is for brevity postponed to another work. It would be also interesting to find some general connection with the topological argument developed in [2].

Full details about the proof of Theorem 1 are contained in the forthcoming paper [8].

References