Combinatorics/Algebra

Are the hyperharmonics integral? A partial answer via the small intervals containing primes

Les hyperharmoniques sont-ils entiers? Une réponse partielle via les petits intervalles contenant des nombres premiers

Rachid Aït Amrane\textsuperscript{a}, Hacène Belbachir\textsuperscript{b}

\textsuperscript{a} ESI/École nationale supérieure d'informatique, BP 68M, Oued Smar, 16309, El Harrach, Alger, Algeria
\textsuperscript{b} USTHB, faculté de mathématiques, BP 32, El Alia, 16111 Bab Ezzouar, Alger, Algeria

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\textbf{ABSTRACT}

In a recent work, the authors have used Bertrand’s postulate to give a partial answer to the conjecture of Mezô which says that the hyperharmonic numbers – iterations of partial sums of harmonic numbers – are not integers. In this Note, using small intervals containing prime numbers, we prove that a great class of hyperharmonic numbers are not integers.

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\textbf{RÉSUMÉ}

Dans un travail antérieur, les auteurs ont utilisé le postulat de Bertrand pour répondre, partiellement, à la conjecture de Mezô selon laquelle les nombres hyperharmoniques – itérations de sommes partielles de nombres harmoniques – ne sont pas des entiers. Dans cette Note, nous montrons qu’une grande classe de nombres hyperharmoniques ne sont pas des entiers en utilisant les petits intervalles contenant des nombres premiers.

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1. Introduction

In 1915, Taesinger proved that the harmonic number $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, except $H_1$, is never an integer (see [10]). In [3, pp. 258–259], Conway and Guy introduced, for a positive integer $r$, the hyperharmonic numbers by the inductive relation:

$$H_n^{(1)} := H_n \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)} \quad (r > 1),$$

where $H_n^{(r)}$ is called the $n$th hyperharmonic number of order $r$, this number can also be expressed in terms of binomial coefficients as follows, see [3],

$$H_n^{(r)} = \binom{n+r-1}{r-1} (H_{n+r-1} - H_{r-1}). \quad (1)$$

E-mail addresses: r.ait.amrane@esi.dz, raitamrane@gmail.com (R. Aït Amrane), hbelbachir@usthb.dz, hacenebelbachir@gmail.com (H. Belbachir).
1. Mezô, in [7], proved that $H_n^{(r)}$, for $r = 2$ and $3$, except in case $H_1^{(r)}$, is never an integer. In his proof, I. Mezô used the reduction modulo the prime number 2. For $n \geq 2$, he conjectured that $H_n^{(r)}$ is never an integer for $r \geq 4$. The authors, in [1], gave an answer to this conjecture in case $r = 2, 3$ and $4$, and for any $r > 4$ when the integers $n + 1, n + 2, \ldots, n + r - 4$ are not primes. The principal tool used in the proof was Bertrand’s postulate which was conjectured in 1845, and assures the existence of a prime number in the interval $[n, 2n - 2]$ when $n \geq 4$. This conjecture, proved by Chebyshev in 1852, lead to much interest from many mathematicians, their interest was about finding the smallest intervals containing a prime number. More general results, see for example [9,5,8,2], have been proved, others conjectured (essentially using the Riemann hypothesis). In this work, we use some of these results to prove that another class of hyperharmonic numbers are not integers.

This new approach which is about reducing the interval in a multiplicative way, i.e. considering intervals of the form $[x, sx]$, leads us to better results than those we got in our preceding paper [1]. In Section 2, we give the main result establishing the non-integerness of hyperharmonic numbers for a large class of $n$ and $r$. Section 3 is devoted to use results that give effective bounds of intervals that contain prime numbers, this allows us to establish the non-integerness of the hyperharmonic numbers $H_n^{(r)}$ when $5 \leq r \leq 25$.

2. Main result

The basic result of this section is a direct consequence of G. Giordano’s theorem, see [4, Theorem 1]:

**Theorem 2.1.** Let $s \in [1, 2]$, then there exists a real number $x_0$ such that for any $x > x_0$, the interval $[x, sx]$ contains at least one prime number.

In his proof, G. Giordano gave explicitly $x_0$. Here:

$$x_0 = \max \left( e^{2s}, \frac{1}{5} \exp (-b + \frac{2}{b} - 4ac) / 2a \right),$$

where $a = 1 - \frac{1}{s}$, $b = -(1 - \frac{1}{s}) \ln s - 0.008(1 + \frac{1}{s})$ and $c = 0.008 \ln s$.

Now, here is our main result:

**Theorem 2.2.** For any $s \in [1, 2]$, there is a prime number $P_0$ such that for any integers $r$ and $n$ with $5 \leq r \leq (2 - s)P_0 + 2$ and $n \geq P_0$, the hyperharmonic number $H_n^{(r)}$ is never an integer.

**Proof.** Let $s \in [1, 2]$, by Theorem 2.1, there is $x_0$ such that for any $x > x_0$ there exists a prime number $Q$ verifying $x < Q \leq sx$. Denote by $P_0$ the smallest prime number strictly greater than $x_0$. Let $r, n$ be such that $5 \leq r \leq (2 - s)P_0 + 2$ and $n \geq P_0$, and assume $H_n^{(r)} \in \mathbb{N}$. We have

$$H_n^{(r)} = \frac{(n + 1)(n + 2) \cdots (n + r - 1)}{(r - 1)!} \left( \frac{1}{r} + \frac{1}{r + 1} + \cdots + \frac{1}{r + n - 1} \right)$$

$$= \frac{(n + 1)(n + 2) \cdots (n + r - 1)}{(r - 1)!} \left( H_n + \frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{n + r - 1} - H_{r-1} \right).$$

Put

$$E_{r,n} := (r - 1)!H_n^{(r)} + (r - 1)! \left( \frac{n + r - 1}{r} \right) H_{r-1} - (n + 1)(n + 2) \cdots (n + r - 1)(H_{n+r-1} - H_n).$$

Therefore

$$E_{r,n} = (n + 1)(n + 2) \cdots (n + r - 1) \left( \frac{1}{2} + \cdots + \frac{1}{n} \right).$$

Since $H_n^{(r)}$ is integral, $E_{r,n}$ also is. Let $P$ be the greatest prime $\leq n$, then $P \geq P_0$ and multiplying the last equality by $\frac{n!}{p}$ we get:

$$\frac{n!}{p} E_{r,n} = \frac{(n + r - 1)!}{p} \left( 1 + \cdots + \frac{1}{p} + \cdots + \frac{1}{n} \right).$$

Hence

$$\frac{n!}{p} E_{r,n} = \frac{(n + r - 1)!}{p} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{p-1} + \frac{1}{p} + \cdots + \frac{1}{n} \right) = \frac{(n + r - 1)!}{p^2}.$$
Theorem 3.2. \( \frac{n}{j} \) divides one of the factors \((n+1), (n+2), \ldots, (n+r-1)\), hence there exists an integer \( j \) such that \( 1 \leq j \leq r-1 \) and \( n+j = 2P \) (more precisely \( j \geq 4 \)). By the hypothesis we have
\[
\frac{n}{j} = (2-s)P_0 + 2 \quad \Rightarrow \quad j \leq r-1 \leq (2-s)P_0 + 1 \leq (2-s)P + 1
\]
\[
\Rightarrow \quad sP \leq 2P - j + 1 = n + 1
\]
\[
\Rightarrow \quad sP \leq n + 1
\]
hence the existence of a prime \( Q \) such that \( x_0 < P < Q \leq sP < n + 1 \), and this contradicts the fact that \( P \) is the greatest integer less than or equal to \( n \). \( \square \)

3. Effective bounds

Observe that the real number \( x_0 \) given in Theorem 2.1 is very large, a thing which explains the particular use of results that give effective bounds of intervals that contain prime numbers.

The basic result for this section is the following theorem (see [6]):

**Theorem 3.1.** For any integer \( n \geq 25 \), there is a prime number \( Q \) such that \( n < Q < (1 + \frac{1}{7}n) \).

We deduce the following first result:

**Theorem 3.2.** The hyperharmonic number \( H_n^{(r)} \), \( n \geq 2 \) and \( 5 \leq r \leq 25 \), is not an integer.

**Proof.** For \( n = 2, 3 \) and \( 4 \), see [1].

For \( 5 \leq r \leq 25 \) and \( n \leq 28 \), a computation with Maple allows to check that \( H_n^{(r)} \) is not an integer.

For \( 5 \leq r \leq 25 \) and \( n \geq 29 \), use Theorem 2.2 with \( s = \frac{5}{2} \) and \( P_0 = 29. \)

The same procedure allows to prove the following theorem:

**Theorem 3.3.**

(i) The hyperharmonic number \( H_n^{(r)} \) is not integral when \( n, r \) are integers such that \( 5 \leq r \leq 2.010.761 \) and \( n \geq 2.010.881 \).

(ii) The hyperharmonic number \( H_n^{(r)} \) is not integral for any integers \( r, n \) such that \( 5 \leq r \leq 10.726.904.664 \) and \( n \geq 10.726.905.041 \).

**Proof.** The results in this theorem use the following:

(i) For any \( n \geq 2.010.760 \), there is a prime \( Q \) such that \( n < Q < (1 + \frac{1}{10077})n \), see [9, Theorem 12].

(ii) For any \( n \geq 10.726.905.041 \), there is a prime number \( Q \) such that \( n < Q < (1 + \frac{1}{28.311.999})n \), see [8, Theorem 3]. \( \square \)

**Remark 1.** In the same context, we can use the table given in [8, p. 11] to show that another infinity of hyperharmonic numbers are not integers.

**Remark 2.** It appears that we can obtain a remarkable improvement using explicit bounds assuming the Riemann Hypothesis. For example, Theorem 1 of [8] says that for \( x \geq 2 \) the interval \( [x - \frac{5}{2}, x] \) contains a prime number.

**References**


