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Partial Differential Equations

On the regularity of capillary water waves with vorticity

Sur la regularité des ondes progressive périodiques avec vorticité

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ABSTRACT

We study the regularity of the streamlines and of the free-surface in a capillary water flow with underlying currents.

RÉSUMÉ

Nous étudions la regularité des lignes de courant et des ondes progressive périodiques avec tension superficielle pour le problème d'écoulement rotationnel de l'eau à surface libre.

1. Introduction

We present some results regarding the regularity of the streamlines and of the free-surface for steady periodic capillary water waves, in the absence of stagnation points, but allowing for rotational flow. Capillary water waves are waves where surface tension is the main restoration force. We make the important assumption that there are no stagnation points, which means that the wave speed is larger than horizontal speed of the individual particles—this assumption is physically plausible for many waves, especially those not near breaking [11,13].

We assume that the water is incompressible and it flows over a flat bed, and we can reformulate the governing equations for periodic steady capillary water waves in terms of the height function $h(q,p)=y+d$ in the fixed rectangular domain $R=\{(q,p)\colon\ q\in\mathbb{R},\ \ p\in[-m,0]\}$ as follows [3,7,16]:

$$\big(1+h_q^2\big)h_{pp}-2h_qh_ph_{pq}+h_p^2h_{qq}-\gamma(-p)h_p^3=0 \quad \text{in } -m<p<0, \tag{1}$$

$$1+h_q^2-Q\,h_p^2-2\sigma\frac{h_p^2h_{qq}}{(1+h_q^2)^{3/2}}=0, \quad p=0, \tag{2}$$

$$h=0, \qquad p=-m. \tag{3}$$

Here $q=x$, the horizontal coordinate, and $p=-\psi$, where ψ is the stream function. The assumption of no stagnation points can be expressed in this setting as $\inf_{(q,p)\in R} h_p(q,p)>0$, which also ensures that the system (1)–(3) is uniformly elliptic. For $\gamma\in C^{1,\alpha}([-m,0])$ the existence of a continuum of solutions $h\in C^{3,\alpha}_{per}(\overline{R})$ was recently proven in [17].

Here the level sets of ψ are the streamlines, and we choose $p=0$ on the free-surface giving us $p=-m$ on the flat bed, where $-m$ equals the relative mass flux. Therefore the regularity of the streamlines (where we consider the free-surface η

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to be a streamline for $\psi = 0$) can be considered by examining the regularity of the mapping

$$q \mapsto h(q, p_0), \quad \text{for fixed } p_0 \in [-m, 0].$$

In the first stages of wind-generated waves, capillary water waves are produced. The effects of surface tension are extremely important for these waves of small amplitude, and they wrinkle the sea much more than larger waves [11]. In the case where the wind has a large enough fetch and blows both long enough and strong enough these smaller waves grow larger and subsequently large amplitude water waves are generated, such as swell in the ocean. In this sense capillary water waves are vital in the study of wave motion. Interestingly, Crapper [5] and Kinnersley [10] have shown the existence of explicit capillary solutions for irrotational water waves of infinite depth, finite depth respectively. In both cases all streamlines are analytic.

Results concerning whether solutions for equations arising in mathematical physics are analytic or not in a certain region are very important—if we know a function is analytic somewhere then we can approximate it to any level of accuracy by using power series, see for example [8]. In 1952 a famous result of Lewy [12] for irrotational water waves showed that a free-surface which is initially C^1 near a point will automatically be analytic near the point. More recently, it was shown that a similar result holds when one takes into account the effects of surface tension [4,14] for irrotational flow, using results from [1,9]. New work in [2] recently proved regularity for steady periodic gravity water waves, with no stagnation points. This work relied on results of [1,9] to show real analyticity of the surface profile, but managed to prove analyticity of the streamlines beneath the wave, for vorticity in $C^{1,\alpha}$, by employing a novel weighted translation to examine the parameter dependence of the nonlinear operator which re-expressed the water wave problem (1)–(3) above. This work generalised the result of Lewy to flows with vorticity, and there have recently been extensions of this result on the streamlines for capillary-gravity waves [6], and for flows whose vorticity is merely bounded and measurable [15].

2. Main results

The following are the three main new results. The first result shows that once the vorticity function $\gamma \in C^{1,\alpha}([-m, 0])$, thereby guaranteeing classical solutions $h \in C^{3,\alpha}_{per}(\overline{R})$, then the mapping $q \mapsto h(q, p_0)$ is smooth for all fixed $p_0 \in [-m, 0]$.

Theorem 2.1. *Let $\gamma \in C^{1,\alpha}([-m, 0])$ and consider the corresponding solution $h \in C^{3,\alpha}_{per}(\overline{R})$ of the governing equations, representing a periodic travelling capillary water wave such that the wave speed exceeds the horizontal fluid velocity throughout the flow. Then each streamline beneath the wave profile, along with the wave profile itself, possesses C^∞ regularity.*

A rough sketch of the proof is given as follows. Upon taking the derivative of the system (1)–(3) with respect to q, and showing that the resulting system is uniformly elliptic in $w = h_q$, we show that the resulting boundary condition satisfies a complementing condition, and the proof follows using results from [1,9].

The next result employs a similar proof as above to show that if the vorticity is analytic, which we denote by $\gamma \in C^\omega([-m, 0])$, then the function $h(q, p)$ is also analytic. We state this result in terms of the regularity of streamlines as follows:

Theorem 2.2. *Let $\gamma \in C^\omega([-m, 0])$ and consider the corresponding solution $h \in C^{3,\alpha}_{per}(\overline{R})$ of the governing equations, representing a periodic travelling capillary water wave such that the wave speed exceeds the horizontal fluid velocity throughout the flow. Then each streamline beneath the wave profile, along with the wave profile itself, is an analytic curve.*

Our final result assures us that we can relax the assumptions on the vorticity from the previous theorem (namely $\gamma \in C^{1,\alpha}([-m, 0])$) to still ensure that all streamlines beneath the free-surface are analytic.

Theorem 2.3. *Let $\gamma \in C^{1,\alpha}([-m, 0])$ and consider the corresponding solution $h \in C^{3,\alpha}_{per}(\overline{R})$ of the governing equations, representing a periodic travelling capillary water wave such that the wave speed exceeds the horizontal fluid velocity throughout the flow. Then each streamline beneath the wave profile is a real analytic curve.*

A sketch of the proof is as follows. We formulate the system (1)–(3) in terms of the operator, $\mathcal{F}(h) := (\mathcal{F}_1(h), \mathcal{F}_2(h))$, where

$$\mathcal{F}_1(h) = \left(\left(1 + h_q^2\right)h_{pp} - 2h_q h_p h_{pq} + h_p^2 h_{qq} - \gamma(-p)h_p^3\right),$$

$$\mathcal{F}_2(h) = \left(1 + h_q^2 - Q h_p^2 - 2\sigma \frac{h_p^2 h_{qq}}{(1 + h_q^2)^{3/2}}\right)\Bigg|_{p=0}.$$

This operator is analytic, and using the weighted translation introduced by Constantin and Escher [2], $\tau_a h(q, p) = h(q + ap, p)$ for $a \in \mathbb{R}$ sufficiently small, we show that $\mathcal{F}(\tau_a h) - \tau_a \mathcal{F}(h) = a\mathcal{K}(\tau_a h, a)$ where $\mathcal{K}$ is another analytic operator. Maximum principles and Fredholm properties then show that for a sufficiently large λ the Fréchet derivative of $\Phi(h, a) = \mathcal{F}(h) -$

$a\mathcal{K}(h,a) + (0, \lambda(h_p - h_p^0 - ah_q^0)|_{P=0})$ with respect to h, where h^0 is a solution of the water wave problem, is invertible. The analyticity of the stream function beneath the surface profile follows from an application of the Implicit Function theorem.

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