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A Note on the von Neumann algebra of a Baumslag-Solitar group

Une Note sur l'algèbre de von Neumann d'un groupe de Baumslag–Solitar

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ARTICLE INFO	ABSTRACT
Article history: Received 15 November 2010 Accepted after revision 8 December 2010 Available online 24 December 2010	We study qualitative properties of the von Neumann algebra of a Baumslag–Solitar group. Namely, we prove that, in the non-amenable and ICC case, the associated II ₁ factor is prime, not solid, and does not have any Cartan subalgebra. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
Presented by the Editorial Board	R É S U M É
	Nous étudions des propriétés qualitatives de l'algèbre de von Neumann d'un groupe de Baumslag–Solitar. À savoir, nous démontrons que, dans le cas non-moyennable et C.C.I., le facteur II ₁ associé est premier, n'est pas solide, et n'a pas de sous-algèbre de Cartan.

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1. Introduction

In their breakthrough paper [9] Ozawa and Popa showed that $L(\mathbb{F}_n)$ is strongly solid. This strengthened two well-known results: Voiculescu's result [14] showing that $L(\mathbb{F}_n)$ has no Cartan subalgebra and Ozawa's result [8] showing that $L(\mathbb{F}_n)$ is solid, which itself strengthened Ge's result [4] showing that $L(\mathbb{F}_n)$ is prime. In this Note we study these properties for Baumslag–Solitar groups factors.

Let $n, m \in \mathbb{Z} - \{0\}$. The Baumslag–Solitar group is defined by $BS(n, m) := \langle a, b | ab^n a^{-1} = b^m \rangle$. It was proved in [7] that $BS(n, m) \simeq BS(p, q)$ if and only if $\{n, m\} = \{\epsilon p, \epsilon q\}$ for some $\epsilon \in \{-1, 1\}$. Moreover, Γ is known to be non-amenable but inner-amenable and ICC whenever $|n|, |m| \ge 2$ and $|n| \ne |m|$ (see [13]). Gal and Januszkiewicz [3] proved that BS(n, m) has the Haagerup property. Note that their proof also implies that BS(n, m) has the complete approximation property (CMAP). Actually one just has to check that the automorphism group of a locally finite tree has the CMAP as a locally compact group (for the compact-open topology). Also, the first L^2 Betti number of BS(n, m) is zero.

Our results can be summarized as follows.

Theorem 1.1. Let $\Gamma = BS(n, m)$. Assume $|n|, |m| \ge 2$ and $|n| \ne |m|$. The following holds:

(i) $L(\Gamma)$ is a prime II₁ factor.

(ii) $L(\Gamma)$ is not solid and does not have any Cartan subalgebra.

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We prove actually a general primeness result for groups acting on trees (see Corollary 3.2). We also prove a stronger property than the absence of Cartan subalgebra (see Theorem 4.2). Namely, we prove that $L(\Gamma)$ is *robust*, i.e. the relative commutant of any regular and amenable von Neumann subalgebra is non-amenable.

2. Preliminaries

2.1. Weakly compact actions

Weakly compact actions were introduced by Ozawa and Popa [9]. The following theorem is similar to [9, Theorem 4.9]. The main ingredients of the proof are contained in the proofs of [9, Theorem 4.9] and [10, Theorem B] as explained in the proof of [6, Theorem 3.5] (see also [5, Theorem 3.3]). This result is not stated explicitly in any of these papers but the proof is the same as the one of [6, Theorem 3.5].

Theorem 2.1. Let *P* be a tracial von Neumann algebra that admits the following deformation property: there exists a tracial von Neumann algebra \tilde{P} , a trace preserving inclusion $P \subset \tilde{P}$ and a one-parameter group $(\alpha_s)_{s \in \mathbb{R}}$ of trace-preserving automorphisms of \tilde{P} such that

- lim_{s→0} $\|\alpha_s(x) x\|_2 = 0$ for all *x* ∈ *P*.
- ${}_{P}L^{2}(\widetilde{P}) \ominus L^{2}(P)_{P}$ is weakly contained in ${}_{P}L^{2}(P) \otimes L^{2}(P)_{P}$.
- There exists c > 0 such that $\|\alpha_{2s}(x) x\|_2 \leq c \|\alpha_s(x) E_P \circ \alpha_s(x)\|_2$, for all $x \in P$, $s \in \mathbb{R}$.

Let $Q \subset P$ be a von Neumann subalgebra and $G \subset \mathcal{N}_P(Q)$ be a subgroup such that the action $G \curvearrowright Q$ is weakly compact. If, for all non-zero projection z in $\mathcal{Z}(G' \cap P)$, α_s does not converge uniformly on $(zQ)_1$, then G'' is amenable.

2.2. HNN extensions of von Neumann algebras

In this Note we follow the approach of [2] for HNN extensions of von Neumann algebras. Let (M, τ_M) be a tracial von Neumann algebra and $N \subset M$ a von Neumann subalgebra. Let $\theta : N \to M$ be trace-preserving embedding. Let $P = HNN(M, N, \theta)$ be the HNN extension. We recall that a tracial von Neumann algebra \tilde{P} with a trace preserving inclusion $P \subset \tilde{P}$ and one parameter group of automorphisms (α_s) of \tilde{P} satisfying the first condition of Theorem 2.1 were constructed in [2]. It was also observed that $\tilde{P} = P *_N (N \otimes L\mathbb{Z})$. This implies that, when N is amenable, $PL^2(\tilde{P}) \ominus L^2(P)_P$ is weakly contained in the coarse P-P-bimodule. A detailed argument can be found, e.g. in [1, Proposition 3.1]. Also, an automorphism $\beta \in Aut(P)$ such that $\beta \circ \alpha_s = \alpha_{-s} \circ \beta$ and $\beta(x) = x$ for all $x \in P$ was introduced in [2]. Such a deformation is *s*-malleable. As such, it automatically satisfies the following transversality property (see [12, Lemma 2.1]): $||\alpha_{2s}(x) - x||_2 \leq 2||\alpha_s(x) - E_M \circ \alpha_s(x)||_2$ for all $x \in M, s \in \mathbb{R}$. Hence, if N is amenable, Theorem 2.1 applies to the HNN extension $P = HNN(M, N, \theta)$.

3. Primeness results for groups acting on trees

The proof of the following proposition is similar to the one of [1, Theorem 5.2] (even easier because we state it in the finite case).

Proposition 3.1. Let M_1 and M_2 be finite von Neumann algebras with a common von Neumann subalgebra B of type I. Let $M = M_1 *_B M_2$. Let $p \in M$ be a non-zero projection. If pMp is a non-amenable II_1 factor then pMp is prime.

The following result is a direct corollary of the preceding proposition and [2, Remark 4.6].

Corollary 3.1. Let $\Gamma = \text{HNN}(H, \Sigma, \theta)$ be a non-trivial HNN extension (i.e. $\Sigma, \theta(\Sigma) \neq H$). Assume Γ is non-amenable and ICC and Σ is abelian or finite. Then, $L(\Gamma)$ is a prime II_1 factor.

Using Corollary 3.1, [1, Theorem 5.2] and arguing as in the proof of [2, Theorem 1.2] we obtain the following result.

Corollary 3.2. Let Γ be non-amenable and ICC group satisfying the following property: Γ admits an action $\Gamma \curvearrowright \mathcal{T}$ without inversion on a tree \mathcal{T} such that there exists a finite subtree with a finite stabilizer and such that there exists an edge $e \in E(\mathcal{T})$ with the properties that Stab *e* is abelian or finite and that the smallest subtrees containing all vertices $\Gamma \cdot s(e)$, resp. $\Gamma \cdot r(e)$, are both equal to the whole of \mathcal{T} . Then, $L(\Gamma)$ is a prime II_1 factor.

4. Robustness for certain HNN extensions

We call a von Neumann algebra *robust* if the relative commutant of any regular and amenable von Neumann subalgebra is non-amenable. Clearly, robustness implies the absence of Cartan subalgebra.

Let (P, τ) be a tracial von Neumann algebra and $A, B \subset P$ be possibly non-unital von Neumann subalgebras. We write $A \prec_P B$ when A embeds in B inside P (i.e. when $1_A L^2(P) 1_B$ admits an A-B-subbimodule H with dim $(H_B) < \infty$, see [11, Section 2]). Otherwise we write $A \not\prec_P B$.

Lemma 4.1. Let $P = \text{HNN}(M, N, \theta)$ be an HNN extension of finite von Neumann algebras and suppose that N is amenable and P has the CMAP. Let $Q \subset P$ a unital von Neumann subalgebra. If Q is amenable and $Q \not\prec_P M$ then $\mathcal{N}_P(Q)''$ is amenable.

Proof. Let $z \in \mathcal{N}_P(Q)' \cap P$ be a non-zero projection. Observe that $z \in Q' \cap P$. As $Q \not\prec_P M$ we get $zQ \not\prec_P M$. By [2, Theorem 3.4] we get that the deformation (α_s) does not converge uniformly on the unit ball of zQ, for all non-zero projection $z \in \mathcal{N}_P(Q)' \cap P$. We can apply [9, Theorem 3.5] and Theorem 2.1 to conclude that $\mathcal{N}_P(Q)''$ is amenable. \Box

Theorem 4.2. Let $\Gamma = \text{HNN}(H, \Sigma, \theta)$. Suppose that the following conditions are satisfied:

(i) *H* is abelian, $2 \leq |H/\Sigma| < \infty$ and $3 \leq |H/\theta(\Sigma)|$,

(ii) Γ has the CMAP.

Then $L(\Gamma)$ is robust. If moreover Σ is infinite, then $L(\Gamma)$ is not solid.

Proof. Let $\Gamma = \langle H, t | \theta(\sigma) = t\sigma t^{-1} \ \forall \sigma \in \Sigma \rangle$. Define $G = \langle H, t^{-1}Ht \rangle \subset \Gamma$ and $\Sigma' = \{g \in \Gamma | g\sigma = \sigma g \text{ for all } \sigma \in \Sigma \}$. As H is abelian, we have $H \subset \Sigma'$. Moreover, for all $\sigma \in \Sigma$ and $h \in H$, we have $t^{-1}ht\sigma = t^{-1}h\theta(\sigma)t = t^{-1}\theta(\sigma)ht = \sigma t^{-1}ht$. It follows that $t^{-1}Ht \subset \Sigma'$. We conclude that $G \subset \Sigma'$.

Let \widetilde{H} be a copy of H and view Σ as a subgroup of \widetilde{H} via the map θ . Define the following group homomorphisms: the first one from H to G is the identity, the second one from \widetilde{H} to G maps h onto $t^{-1}ht$. Homomorphisms of these groups agree on Σ (because we see $\Sigma < \widetilde{H}$ via the map θ). We get a group homomorphism from $H *_{\Sigma} \widetilde{H}$ to G which is clearly surjective. It is also injective because it maps each reduced word in $H *_{\Sigma} \widetilde{H}$ onto a reduced word (in the HNN extension sense) in G. As $|H/\Sigma| \ge 2$ and $|H/\theta(\Sigma)| \ge 3$, $H *_{\Sigma} \widetilde{H}$ is not amenable.

Let $Q \subset L(\Gamma)$ be an amenable regular subalgebra. By Lemma 4.1, $Q \prec_{L(\Gamma)} L(H)$. Because Σ has finite index in H we obtain $Q \prec_{L(\Gamma)} L(\Sigma)$. It follows that $L(\Sigma)' \cap L(\Gamma) \prec_{L(\Gamma)} Q' \cap M$. In particular, $L(G) \prec_{L(\Gamma)} Q' \cap M$. Because L(G) has no amenable direct summand, $Q' \cap M$ is not amenable. If $L(\Sigma)$ is infinite, $L(\Gamma)$ is obviously not solid. Actually, $L(\Sigma)$ is a diffuse amenable von Neumann subalgebra and its relative commutant is not amenable as it contains L(G). \Box

We obtain the following obvious corollary.

Corollary 4.3. Let $\Gamma = BS(m, n)$. If $|n|, |m| \ge 2$ and $|n| \ne |m|$ then $L(\Gamma)$ is a non-solid II_1 factor and does not have any Cartan subalgebra.

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