# Asymptotic analysis for a diffusion problem 

## Analyse asymptotique pour un problème de diffusion

Khaled El-Ghaouti Boutarene<br>AMNEDP Laboratory, Faculty of Mathematics, USTHB, Po Box 32, El Alia 16111, Babezzouar, Algiers, Algeria

## A R T I C L E IN F O

## Article history:

Received 15 July 2010
Accepted 6 December 2010
Available online 23 December 2010
Presented by Jean-Pierre Demailly


#### Abstract

This Note describes a method for deriving an asymptotic expansion of the solution of Laplace equation in a bounded domain of $\mathbb{R}^{P}(P=2,3)$. This domain is composed of two subdomains and a separating thin layer of thickness $\delta$ (destined to tend to 0 ). The method is based on hierarchical variational equations which are suitable for the construction of the asymptotic expansion up to any order.


© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## R É S U M É

Dans cette Note, nous présentons une méthode pour construire un développement asymptotique de la solution de l'équation de Laplace dans un domaine borné de $\mathbb{R}^{P}$ ( $P=2,3$ ). Ce domaine est composé de deux sous-domaines séparés par une couche mince d'épaisseur $\delta$ (destinée à tendre vers 0 ). La méthode est basée sur une hiérarchie d'équations variationnelles qui se prêtent au calcul du développement asymptotique à tout ordre.
© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## 1. Introduction

The aim of this work is to study the asymptotic behavior of the solution of Laplace equation in a bounded domain $\Omega$ of $\mathbb{R}^{P}(P=2,3)$ consisting of three subdomains: an open bounded subset $\Omega_{i}$ with regular boundary $\Gamma$, an exterior domain $\Omega_{e}$ with disjoint regular boundaries $\Gamma_{\delta}$ and $\partial \Omega$, and a membrane $\Omega_{\delta}$ (thin layer) of thickness $\delta$ separating $\Omega_{i}$ from $\Omega_{e}$ (see Fig. 1). We are interested in the following problem:

$$
\left\{\begin{array}{llll}
\Delta u_{e, \delta}=0 & \text { in } \Omega_{e}, & u_{d, \delta}=u_{e, \delta} & \text { on } \Gamma_{\delta},  \tag{1}\\
\alpha \Delta u_{d, \delta}=0 & \text { in } \Omega_{\delta}, & \alpha \partial_{\mathbf{n}} u_{d, \delta}=\partial_{\mathbf{n}} u_{e, \delta} & \text { on } \Gamma_{\delta}, \\
\beta \Delta u_{i, \delta}=-f_{i} & \text { in } \Omega_{i}, & u_{i, \delta}=u_{d, \delta} & \text { on } \Gamma, \\
u_{e, \delta}=0 & \text { on } \partial \Omega, & \beta \partial_{\mathbf{n}} u_{i, \delta}=\alpha \partial_{\mathbf{n}} u_{d, \delta} & \text { on } \Gamma,
\end{array}\right.
$$

where $\partial_{\mathbf{n}}$ denotes the normal derivative (outward to $\left.\Omega_{i}\right), \alpha$ and $\beta$ are some positive constants and $f_{i} \in \mathcal{C}^{\infty}\left(\bar{\Omega}_{i}\right)$.
The solution of this problem via finite element methods exhibits numerical instabilities when the thickness $\delta$ of the layer is considerably small than the size of neighboring (cf. [6]). To avoid this difficulty, we perform asymptotic analysis to model the effect of the thin layer by conditions on the interface $\Gamma$.

[^0]

Fig. 1. Geometric data.
Poignard, Schmidt and Tordeux have worked on similar problems in [4,5] and [6]. They studied Helmholtz equation in the bidimensional case with different boundary conditions. They gave an expansion of Laplace operator in a fixed domain (independent of $\delta$ ) and obtained an asymptotic expansion of the solution of Helmholtz equation with appropriate transmission conditions.

In the present paper, a new framework is proposed with a model problem. It is based on variational formulations (cf. [1]) which allow one to derive asymptotic expansion up to any order in a simple way. The cases 3D and 2D are similar, we treat the three-dimensional case, and 2D comes in Remark 1.

The present work is organized as follows: In Section 2, we set the notations and definitions from differential geometry of surfaces [2] (see also [1] and [3]), which are useful to the following theoretical developments. Section 3 is devoted to the asymptotic analysis of our system by giving a formal asymptotic expansion of the solution of problem (1). We determine the first three terms of the expansion and we establish a convergence theorem related to the justification up to any order of the ansatz.

## 2. Definitions and notations

Let $(\mathcal{U}, \varphi)$ be a local coordinate patch for the surface $\Gamma$, with $\mathcal{U}$ being an open domain of $\mathbb{R}^{2}$ and $\varphi: \mathcal{U} \rightarrow \Gamma$ such that $\varphi\left(\xi^{1}, \xi^{2}\right)=m$. We parameterize the thin shell $\Omega_{\delta}$ by the manifold $\Omega^{+}=\Gamma \times(0,1)$ through the mapping $\eta$ defined by $\eta: \Omega^{+} \rightarrow \Omega_{\delta}$ such that $\eta(m, s)=m+\delta \mathbf{s n}(m)=\varphi\left(\xi^{1}, \xi^{2}\right)+\delta \operatorname{sn}\left(\varphi\left(\xi^{1}, \xi^{2}\right)\right)$.

To each function $v$ defined on $\Omega_{\delta}$, we associate the function $V^{+}$defined on $\Omega^{+}$by $V^{+}(m, s):=v \circ \eta(m, s)$. Let $U^{+}$and $V^{+}$be two regular functions in $H^{1}\left(\Omega^{+}\right)$. We define the bilinear form $a^{+}(\delta ;,$.$) (cf. [1]) by$

$$
\begin{align*}
\delta a^{+}\left(\delta ; U^{+}, V^{+}\right):= & \alpha \delta^{-1} \int_{\Gamma} \int_{0}^{1} \partial_{s} U^{+} \partial_{s} V^{+} \operatorname{det}(I+s \delta \mathcal{R}) \mathrm{d} s \mathrm{~d} \Gamma \\
& +\alpha \delta \int_{\Gamma} \int_{0}^{1}(I+s \delta \mathcal{R})^{-2} \nabla_{\Gamma} U^{+} . \nabla_{\Gamma} V^{+} \operatorname{det}(I+s \delta \mathcal{R}) \mathrm{d} s \mathrm{~d} \Gamma=\alpha \int_{\Omega_{\delta}} \nabla u . \nabla v \mathrm{~d} \Omega_{\delta}, \tag{2}
\end{align*}
$$

where $\mathcal{R}$ is the symmetric linear operator of the tangent plane $T_{m}(\Gamma)$ that characterizes the curvature of $\Gamma$ at point $m$ and $\nabla_{\Gamma} v(m)$ is the surface gradient of $v$ at $m \in \Gamma$. Finally, we denote by $\mathcal{H}$ and $\mathcal{K}$ the mean and the Gaussian curvatures of the surface $\Gamma$ respectively.

## 3. The asymptotic analysis

Let $v_{d}$ be a regular function in $H^{1}\left(\Omega_{\delta}\right)$. We multiply $\Delta u_{d, \delta}$ by $v_{d}$, using Green formula and transmission condition of problem (1), we obtain

$$
\begin{equation*}
\int_{\Gamma} \beta \partial_{n} u_{i, \delta} v_{d / \Gamma} \mathrm{d} \Gamma+\alpha \int_{\Omega_{\delta}} \nabla u_{d, \delta} . \nabla v_{d} \mathrm{~d} \Omega_{\delta}+\int_{\Gamma_{\delta}} \partial_{n} u_{e, \delta} v_{d / \Gamma_{\delta}} \mathrm{d} \Gamma_{\delta}=0 \tag{3}
\end{equation*}
$$

We remember that $U_{d, \delta}^{+}:=u_{d, \delta} \circ \eta$; in a natural way, we consider the following ansatz

$$
\begin{equation*}
u_{i, \delta}=\sum_{n \geqslant 0} \delta^{n} u_{i, n} \quad \text { in } \Omega_{i}, \quad U_{d, \delta}^{+}=\sum_{n \geqslant 0} \delta^{n} U_{n}^{+} \quad \text { in } \Gamma \times[0,1] \quad \text { and } \quad u_{e, \delta}=\sum_{n \geqslant 0} \delta^{n} u_{e, n} \quad \text { in } \Omega_{e}, \tag{4}
\end{equation*}
$$

where the terms $u_{i, n}, U_{n}^{+}$and $u_{e, n}$ are independent of $\delta$. As in [4], we extend formally $u_{e, \delta}$ to $\Omega \backslash \Omega_{i}$, by extending a finite number of coefficients of the power $\delta$. A Taylor expansion gives

$$
\begin{equation*}
\partial_{\mathbf{n}} u_{e, \delta} \circ \eta(m, s)=\partial_{\mathbf{n}} u_{e, 0 / \Gamma} \circ \eta(m, 0)+\delta\left[\partial_{\mathbf{n}} u_{e, 1 / \Gamma} \circ \eta(m, 0)+s \partial_{\mathbf{n}}^{2} u_{e, 0 / \Gamma} \circ \eta(m, 0)\right]+\cdots . \tag{5}
\end{equation*}
$$

Inserting the asymptotic expansions (4) and (5) into the variational equation (3) we obtain, for each function $V^{+} \in H^{1}\left(\Omega^{+}\right)$,

$$
\begin{align*}
& \int_{\Gamma} \beta\left\{\left(\sum_{n \geqslant 0} \delta^{n} \partial_{\mathbf{n}} u_{i, n / \Gamma}\right) \circ \eta(m, 0)\right\} V^{+}(m, 0) \mathrm{d} \Gamma+\delta a^{+}\left(\delta ; \sum_{n \geqslant 0} \delta^{n} U_{n}^{+}, V^{+}\right)-\int_{\Gamma}\left\{\partial_{\mathbf{n}} u_{e, 0 / \Gamma} \circ \eta(m, 0)\right. \\
& \left.\quad+\delta\left[\partial_{\mathbf{n}} u_{e, 1 / \Gamma} \circ \eta(m, 0)+\partial_{\mathbf{n}}^{2} u_{e, 0 / \Gamma} \circ \eta(m, 0)\right]+\cdots\right\} V^{+}(m, 1) \operatorname{det}(I+\delta \mathcal{R}) \mathrm{d} \Gamma=0 \tag{6}
\end{align*}
$$

In order to calculate the terms $u_{i, n}, u_{e, n}$ and $U_{n}^{+}$, we give an expansion of the bilinear form $a^{+}(\delta ; .,$.$) in powers of \delta$ (cf. [1]), inserting it into (6) and matching the same power of $\delta$, we obtain a hierarchy of variational equations. The first three terms of the asymptotic expansion are given by solving the following problems

$$
\begin{equation*}
\beta \Delta u_{i, n}=-f_{i} \delta_{0}^{n} \quad \text { in } \Omega_{i}, \quad \Delta u_{e, n}=0 \quad \text { in } \Omega \backslash \bar{\Omega}_{i}, \quad u_{e, n}=0 \quad \text { on } \partial \Omega, n \leqslant 2, \tag{7}
\end{equation*}
$$

where $\delta_{0}^{n}$ is the Kronecker symbol, with transmission conditions

- of order $0:\left\{\begin{array}{l}u_{i, 0 / \Gamma}-u_{e, 0 / \Gamma}=0, \\ \beta \partial_{\mathbf{n}} u_{i, 0 / \Gamma}=\partial_{\mathbf{n}} u_{e, 0 / \Gamma},\end{array}\right.$
- of order 1: $\left\{\begin{array}{l}u_{i, 1 / \Gamma}-u_{e, 1 / \Gamma}=\left(1-\frac{1}{\alpha}\right) \partial_{\mathbf{n}} u_{e, 0 / \Gamma}, \\ \beta \partial_{\mathbf{n}} u_{i, 1 / \Gamma}-\partial_{\mathbf{n}} u_{e, 1 / \Gamma}=2 \mathcal{H} \partial_{\mathbf{n}} u_{e, 0 / \Gamma}+\partial_{\mathbf{n}}^{2} u_{e, 0 / \Gamma}+\alpha \Delta_{\Gamma} u_{i, 0 / \Gamma},\end{array}\right.$

The terms $U_{n}^{+}, n \leqslant 2$, are given by

$$
\begin{aligned}
U_{0}^{+}(m, s)= & u_{i, 0 / \Gamma} \circ \eta(m, 0)=u_{e, 0 / \Gamma} \circ \eta(m, 0), \\
U_{1}^{+}(m, s)= & u_{i, 1 / \Gamma} \circ \eta(m, 0)+\frac{s}{\alpha} \partial_{\mathbf{n}} u_{e, 0 / \Gamma} \circ \eta(m, 0), \\
U_{2}^{+}(m, s)= & u_{i, 2 / \Gamma} \circ \eta(m, 0)+\frac{\mathcal{H}}{\alpha}\left(2 s-s^{2}\right) \partial_{\mathbf{n}} u_{e, 0 / \Gamma} \circ \eta(m, 0)+\frac{s}{\alpha} \partial_{\mathbf{n}} u_{e, 1 / \Gamma} \circ \eta(m, 0) \\
& +\frac{s}{\alpha} \partial_{\mathbf{n}}^{2} u_{e, 0 / \Gamma} \circ \eta(m, 0)+\left(s-\frac{s^{2}}{2}\right) \Delta_{\Gamma} u_{i, 0 / \Gamma} \circ \eta(m, 0),
\end{aligned}
$$

for all $(m, s) \in \Gamma \times[0,1]$.
Remark 1. The determination of the terms of the asymptotic expansion in the two-dimensional case does not differ from the case $P=3$. It suffices to replace $\mathcal{K}$ by 0 and $2 \mathcal{H}$ by $\mathcal{R}$.

We can also estimate the error made by truncating the series (4) after a finite number of terms. Let

$$
u_{i, \delta}^{N}:=\sum_{n=0}^{n=N} \delta^{n} u_{i, n}, \quad u_{e, \delta}^{N}:=\sum_{n=0}^{n=N} \delta^{n} u_{e, n} \quad \text { and } \quad u_{d, \delta}^{N}:=\sum_{n=0}^{n=N} \delta^{n} u_{n}^{+},
$$

where $u_{n}^{+}(m, \delta s):=U_{n}^{+}(m, s) ; \forall(m, s) \in \Gamma \times[0,1]$ and $N \in \mathbb{N}$.
Theorem 3.1. For all integers $N \geqslant 0$, there exists a constant $C$ independent of $\delta$ such as

$$
\left\|u_{i, \delta}-u_{i, \delta}^{N}\right\|_{H^{1}\left(\Omega_{i}\right)}+\sqrt{\delta}\left\|u_{d, \delta}-u_{d, \delta}^{N}\right\|_{H^{1}\left(\Omega_{\delta}\right)}+\left\|u_{e, \delta}-u_{e, \delta}^{N}\right\|_{H^{1}\left(\Omega_{e}\right)} \leqslant C \delta^{N+1} .
$$

## References

[1] A. Bendali, K. Lemrabet, The effect of a thin coating on the scattering of the time-harmonic wave for the Helmholtz equation, SIAM J. Appl. Math. 56 (6) (1996) 1664-1693
[2] M.P. Do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, NJ, 1976.
[3] J.C. Nedelec, Acoustic and Electromagnetic Equations, Integral Representations for Harmonic Problems, Springer, 2001.
[4] C. Poignard, Méthodes asymptotiques pour le calcul des champs électromagnétiques dans des milieux à couches minces. Application aux cellules biologiques, Thèse de Doctorat, Université Claude Bernard-Lyon 1, 2006.
[5] K. Schmidt, High-order numerical modeling of highly conductive thin sheets, PhD thesis, ETH Zurich, 2008.
[6] K. Schmidt, S. Tordeux, Asymptotic modelling of conductive thin sheets, Research Report No. 2008-28, Swiss Federal Institute of Technology Zurich, 2008


[^0]:    E-mail address: boutarenekhaled@yahoo.fr.
    1631-073X/\$ - see front matter © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.
    doi:10.1016/j.crma.2010.12.002

