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Partial Differential Equations

Asymptotic analysis for a diffusion problem

Analyse asymptotique pour un problème de diffusion

Khaled El-Ghaouti Boutarene

AMNEDP Laboratory, Faculty of Mathematics, USTHB, Po Box 32, El Alia 16111, Babezzouar, Algiers, Algeria

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ABSTRACT

This Note describes a method for deriving an asymptotic expansion of the solution of Laplace equation in a bounded domain of \mathbb{R}^P (P = 2, 3). This domain is composed of two subdomains and a separating thin layer of thickness δ (destined to tend to 0). The method is based on hierarchical variational equations which are suitable for the construction of the asymptotic expansion up to any order.

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RÉSUMÉ

Dans cette Note, nous présentons une méthode pour construire un développement asymptotique de la solution de l'équation de Laplace dans un domaine borné de \mathbb{R}^{P} (P = 2, 3). Ce domaine est composé de deux sous-domaines séparés par une couche mince d'épaisseur δ (destinée à tendre vers 0). La méthode est basée sur une hiérarchie d'équations variationnelles qui se prêtent au calcul du développement asymptotique à tout ordre.

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1. Introduction

The aim of this work is to study the asymptotic behavior of the solution of Laplace equation in a bounded domain Ω of \mathbb{R}^{P} (P = 2, 3) consisting of three subdomains: an open bounded subset Ω_{i} with regular boundary Γ , an exterior domain Ω_{e} with disjoint regular boundaries Γ_{δ} and $\partial \Omega$, and a membrane Ω_{δ} (thin layer) of thickness δ separating Ω_{i} from Ω_{e} (see Fig. 1). We are interested in the following problem:

1	$\Delta u_{e,\delta}=0$	in Ω_e ,	$u_{d,\delta} = u_{e,\delta}$	on Γ_{δ} ,	
	$\alpha \Delta u_{d,\delta} = 0$	in Ω_{δ} ,	$\alpha \partial_{\mathbf{n}} u_{d,\delta} = \partial_{\mathbf{n}} u_{e,\delta}$	on Γ_{δ} ,	(1)
ĺ	$\beta \Delta u_{i,\delta} = -f_i$	in Ω_i ,	$u_{i,\delta} = u_{d,\delta}$	on Γ ,	(1)
	$u_{e,\delta}=0$	on $\partial \Omega$,	$\beta \partial_{\mathbf{n}} u_{i,\delta} = \alpha \partial_{\mathbf{n}} u_{d,\delta}$	on Γ ,	

where $\partial_{\mathbf{n}}$ denotes the normal derivative (outward to Ω_i), α and β are some positive constants and $f_i \in C^{\infty}(\overline{\Omega}_i)$.

The solution of this problem via finite element methods exhibits numerical instabilities when the thickness δ of the layer is considerably small than the size of neighboring (cf. [6]). To avoid this difficulty, we perform asymptotic analysis to model the effect of the thin layer by conditions on the interface Γ .

E-mail address: boutarenekhaled@yahoo.fr.

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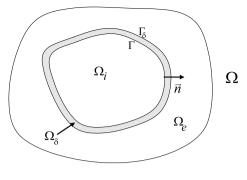


Fig. 1. Geometric data.

Poignard, Schmidt and Tordeux have worked on similar problems in [4,5] and [6]. They studied Helmholtz equation in the bidimensional case with different boundary conditions. They gave an expansion of Laplace operator in a fixed domain (independent of δ) and obtained an asymptotic expansion of the solution of Helmholtz equation with appropriate transmission conditions.

In the present paper, a new framework is proposed with a model problem. It is based on variational formulations (cf. [1]) which allow one to derive asymptotic expansion up to any order in a simple way. The cases 3D and 2D are similar, we treat the three-dimensional case, and 2D comes in Remark 1.

The present work is organized as follows: In Section 2, we set the notations and definitions from differential geometry of surfaces [2] (see also [1] and [3]), which are useful to the following theoretical developments. Section 3 is devoted to the asymptotic analysis of our system by giving a formal asymptotic expansion of the solution of problem (1). We determine the first three terms of the expansion and we establish a convergence theorem related to the justification up to any order of the ansatz.

2. Definitions and notations

Let (\mathcal{U}, φ) be a local coordinate patch for the surface Γ , with \mathcal{U} being an open domain of \mathbb{R}^2 and $\varphi: \mathcal{U} \to \Gamma$ such that $\varphi(\xi^1,\xi^2) = m$. We parameterize the thin shell Ω_{δ} by the manifold $\Omega^+ = \Gamma \times (0,1)$ through the mapping η defined by $\eta: \Omega^+ \to \Omega_{\delta}$ such that $\eta(m,s) = m + \delta s \mathbf{n}(m) = \varphi(\xi^1,\xi^2) + \delta s \mathbf{n}(\varphi(\xi^1,\xi^2))$. To each function ν defined on Ω_{δ} , we associate the function V^+ defined on Ω^+ by $V^+(m,s) := \nu \circ \eta(m,s)$. Let U^+ and

 V^+ be two regular functions in $H^1(\Omega^+)$. We define the bilinear form $a^+(\delta; ..., .)$ (cf. [1]) by

$$\delta a^{+}(\delta; U^{+}, V^{+}) := \alpha \delta^{-1} \int_{\Gamma} \int_{0}^{1} \partial_{s} U^{+} \partial_{s} V^{+} \det(I + s \delta \mathcal{R}) \, \mathrm{d}s \, \mathrm{d}\Gamma$$
$$+ \alpha \delta \int_{\Gamma} \int_{0}^{1} (I + s \delta \mathcal{R})^{-2} \nabla_{\Gamma} U^{+} \cdot \nabla_{\Gamma} V^{+} \det(I + s \delta \mathcal{R}) \, \mathrm{d}s \, \mathrm{d}\Gamma = \alpha \int_{\Omega_{\delta}} \nabla u \cdot \nabla v \, \mathrm{d}\Omega_{\delta}, \tag{2}$$

where \mathcal{R} is the symmetric linear operator of the tangent plane $T_m(\Gamma)$ that characterizes the curvature of Γ at point *m* and $\nabla_{\Gamma} v(m)$ is the surface gradient of v at $m \in \Gamma$. Finally, we denote by \mathcal{H} and \mathcal{K} the mean and the Gaussian curvatures of the surface Γ respectively.

3. The asymptotic analysis

Let v_d be a regular function in $H^1(\Omega_{\delta})$. We multiply $\Delta u_{d,\delta}$ by v_d , using Green formula and transmission condition of problem (1), we obtain

$$\int_{\Gamma} \beta \partial_n u_{i,\delta} v_{d/\Gamma} \, \mathrm{d}\Gamma + \alpha \int_{\Omega_{\delta}} \nabla u_{d,\delta} \cdot \nabla v_d \, \mathrm{d}\Omega_{\delta} + \int_{\Gamma_{\delta}} \partial_n u_{e,\delta} v_{d/\Gamma_{\delta}} \, \mathrm{d}\Gamma_{\delta} = 0.$$
(3)

We remember that $U_{d,\delta}^+ := u_{d,\delta} \circ \eta$; in a natural way, we consider the following ansatz

$$u_{i,\delta} = \sum_{n \ge 0} \delta^n u_{i,n} \quad \text{in } \Omega_i, \qquad U_{d,\delta}^+ = \sum_{n \ge 0} \delta^n U_n^+ \quad \text{in } \Gamma \times [0,1] \quad \text{and} \quad u_{e,\delta} = \sum_{n \ge 0} \delta^n u_{e,n} \quad \text{in } \Omega_e, \tag{4}$$

where the terms $u_{i,n}$, U_n^+ and $u_{e,n}$ are independent of δ . As in [4], we extend formally $u_{e,\delta}$ to $\Omega \setminus \Omega_i$, by extending a finite number of coefficients of the power δ . A Taylor expansion gives

$$\partial_{\mathbf{n}} u_{e,\delta} \circ \eta(m,s) = \partial_{\mathbf{n}} u_{e,0/\Gamma} \circ \eta(m,0) + \delta \left[\partial_{\mathbf{n}} u_{e,1/\Gamma} \circ \eta(m,0) + s \partial_{\mathbf{n}}^2 u_{e,0/\Gamma} \circ \eta(m,0) \right] + \cdots$$
(5)

Inserting the asymptotic expansions (4) and (5) into the variational equation (3) we obtain, for each function $V^+ \in H^1(\Omega^+)$,

$$\int_{\Gamma} \beta \left\{ \left(\sum_{n \ge 0} \delta^n \partial_{\mathbf{n}} u_{i,n/\Gamma} \right) \circ \eta(m,0) \right\} V^+(m,0) \, \mathrm{d}\Gamma + \delta a^+ \left(\delta; \sum_{n \ge 0} \delta^n U_n^+, V^+ \right) - \int_{\Gamma} \left\{ \partial_{\mathbf{n}} u_{e,0/\Gamma} \circ \eta(m,0) + \delta \left[\partial_{\mathbf{n}} u_{e,0/\Gamma} \circ \eta(m,0) \right] + \cdots \right\} V^+(m,1) \, \mathrm{det}(I + \delta \mathcal{R}) \, \mathrm{d}\Gamma = 0.$$
(6)

In order to calculate the terms $u_{i,n}$, $u_{e,n}$ and U_n^+ , we give an expansion of the bilinear form $a^+(\delta;.,.)$ in powers of δ (cf. [1]), inserting it into (6) and matching the same power of δ , we obtain a hierarchy of variational equations. The first three terms of the asymptotic expansion are given by solving the following problems

 $\beta \Delta u_{i,n} = -f_i \delta_0^n \quad \text{in } \Omega_i, \qquad \Delta u_{e,n} = 0 \quad \text{in } \Omega \setminus \bar{\Omega}_i, \qquad u_{e,n} = 0 \quad \text{on } \partial \Omega, \ n \leq 2,$ (7)

where δ_0^n is the Kronecker symbol, with transmission conditions

$$- \text{ of order 0:} \begin{cases} u_{i,0/\Gamma} - u_{e,0/\Gamma} = 0, \\ \beta \partial_{\mathbf{n}} u_{i,0/\Gamma} = \partial_{\mathbf{n}} u_{e,0/\Gamma}, \end{cases}$$

$$- \text{ of order 1:} \begin{cases} u_{i,1/\Gamma} - u_{e,1/\Gamma} = \left(1 - \frac{1}{\alpha}\right) \partial_{\mathbf{n}} u_{e,0/\Gamma}, \\ \beta \partial_{\mathbf{n}} u_{i,1/\Gamma} - \partial_{\mathbf{n}} u_{e,1/\Gamma} = 2\mathcal{H} \partial_{\mathbf{n}} u_{e,0/\Gamma} + \partial_{\mathbf{n}}^{2} u_{e,0/\Gamma} + \alpha \Delta_{\Gamma} u_{i,0/\Gamma}, \end{cases}$$

$$- \text{ of order 2:} \begin{cases} u_{i,2/\Gamma} - u_{e,2/\Gamma} = \left(1 - \frac{1}{\alpha}\right) \partial_{\mathbf{n}} u_{e,1/\Gamma} - \frac{\mathcal{H}}{\alpha} \partial_{\mathbf{n}} u_{e,0/\Gamma} + \left(\frac{1}{2} - \frac{1}{\alpha}\right) \partial_{\mathbf{n}}^{2} u_{e,0/\Gamma} - \frac{1}{2} \Delta_{\Gamma} u_{i,0/\Gamma}, \end{cases}$$

$$+ \frac{1}{2} \partial_{\mathbf{n}}^{3} u_{e,0/\Gamma} + \alpha \Delta_{\Gamma} u_{i,1/\Gamma} + \frac{1}{2} \Delta_{\Gamma} \partial_{\mathbf{n}} u_{e,0/\Gamma} + \alpha \operatorname{div}_{\Gamma} [(\mathcal{H}I - \mathcal{R}) \nabla_{\Gamma} u_{i,0/\Gamma}]. \end{cases}$$

The terms U_n^+ , $n \leq 2$, are given by

$$\begin{split} U_{0}^{+}(m,s) &= u_{i,0/\Gamma} \circ \eta(m,0) = u_{e,0/\Gamma} \circ \eta(m,0), \\ U_{1}^{+}(m,s) &= u_{i,1/\Gamma} \circ \eta(m,0) + \frac{s}{\alpha} \partial_{\mathbf{n}} u_{e,0/\Gamma} \circ \eta(m,0), \\ U_{2}^{+}(m,s) &= u_{i,2/\Gamma} \circ \eta(m,0) + \frac{\mathcal{H}}{\alpha} (2s - s^{2}) \partial_{\mathbf{n}} u_{e,0/\Gamma} \circ \eta(m,0) + \frac{s}{\alpha} \partial_{\mathbf{n}} u_{e,1/\Gamma} \circ \eta(m,0) \\ &+ \frac{s}{\alpha} \partial_{\mathbf{n}}^{2} u_{e,0/\Gamma} \circ \eta(m,0) + \left(s - \frac{s^{2}}{2}\right) \Delta_{\Gamma} u_{i,0/\Gamma} \circ \eta(m,0), \end{split}$$

for all $(m, s) \in \Gamma \times [0, 1]$.

Remark 1. The determination of the terms of the asymptotic expansion in the two-dimensional case does not differ from the case P = 3. It suffices to replace \mathcal{K} by 0 and $2\mathcal{H}$ by \mathcal{R} .

We can also estimate the error made by truncating the series (4) after a finite number of terms. Let

$$u_{i,\delta}^N := \sum_{n=0}^{n=N} \delta^n u_{i,n}, \qquad u_{e,\delta}^N := \sum_{n=0}^{n=N} \delta^n u_{e,n} \quad \text{and} \quad u_{d,\delta}^N := \sum_{n=0}^{n=N} \delta^n u_n^+,$$

where $u_n^+(m, \delta s) := U_n^+(m, s); \ \forall (m, s) \in \Gamma \times [0, 1] \text{ and } N \in \mathbb{N}.$

Theorem 3.1. For all integers $N \ge 0$, there exists a constant *C* independent of δ such as

$$\left\|u_{i,\delta}-u_{i,\delta}^{N}\right\|_{H^{1}(\Omega_{i})}+\sqrt{\delta}\left\|u_{d,\delta}-u_{d,\delta}^{N}\right\|_{H^{1}(\Omega_{\delta})}+\left\|u_{e,\delta}-u_{e,\delta}^{N}\right\|_{H^{1}(\Omega_{e})}\leq C\delta^{N+1}.$$

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