

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I



www.sciencedirect.com

A new quantity in Finsler geometry

Une nouvelle quantité en géométrie finslerienne

Behzad Najafi^a, Akbar Tayebi^b

^a Faculty of Science, Department of Mathematics, Shahed University, Tehran, Iran
 ^b Faculty of Science, Department of Mathematics, Qom University, Qom, Iran

ARTICLE INFO

Differential Geometry

Article history: Received 24 June 2010 Accepted after revision 16 November 2010 Available online 26 November 2010

Presented by Jean-Pierre Demailly

ABSTRACT

In this Note, we define a new quantity and call it C-projective Weyl curvature. We prove that for a Finsler manifold of scalar flag curvature with dimension $n \ge 3$, $\mathbf{H} = 0$ if and only if $\widetilde{W} = 0$.

© 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Dans cette Note, nous définissons une nouvelle quantité que nous appelons courbure Cprojective de Weyl. Nous montrons que pour une variété de Finsler de dimension $n \ge 3$ ayant une courbure de drapeaux de type scalaire, on a $\mathbf{H} = 0$ si et seulement si $\widetilde{W} = 0$. © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

There are several important non-Riemannian quantities such as the Cartan torsion **C**, the Berwald curvature **B**, the mean Berwald curvature **E** and the Landsberg curvature **L**, etc. They all vanish for Riemannian metrics, hence they are said to be non-Riemannian [3]. In [1], Akbar-Zadeh introduces the non-Riemannian quantity **H** which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics [2]. This is a positively homogeneous scalar function of degree zero on the slit tangent bundle, and recently has been studied by X. Mo for a Finsler metric and established a natural relation among **H** and the flag curvature [4]. Akbar-Zadeh proved that for a Finsler metric of scalar flag curvature with dimension $n \ge 3$, the flag curvature is constant if and only if $\mathbf{H} = 0$ [1]. Is there another Finslerian quantity which characterizes Finsler metrics of constant flag curvature?

In this paper, we define a new quantity for Finsler metrics and call it \widetilde{W} -curvature. We show that the \widetilde{W} -curvature is a projective invariant and another candidate for characterizing Finsler metrics of constant flag curvature. More precisely, we prove the following:

Theorem 1. Let (M, F) be a Finsler manifold of scalar flag curvature with dimension $n \ge 3$. Then $\mathbf{H} = 0$ if and only if $\widetilde{W} = 0$.

E-mail addresses: najafi@shahed.ac.ir (B. Najafi), akbar.tayebi@gmail.com (A. Tayebi).

¹⁶³¹⁻⁰⁷³X/\$ – see front matter © 2010 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved. doi:10.1016/j.crma.2010.11.015

2. C-projective Weyl curvature

Let *F* be a Finsler metric on a manifold *M*. The geodesics of *F* are characterized locally by the equation $\ddot{c}^i + 2G^i(\dot{c}) = 0$. For a vector v^i vertical and horizontal covariant derivative with respect to Berwald connection are given by $v^i_{,k} = \dot{\partial}_k v^i$, $v^i_{,k} = d_k v^i + G^i_{jk} v^j$ where $d_k = \partial_k - G^m_k \dot{\partial}_m$, $\partial_k = \frac{\partial}{\partial x^k}$, $\dot{G}^i_k = \dot{\partial}_k G^i$ and $G^i_{jk} = \dot{\partial}_j G^i_k$.

Let $\phi : F^n \to \overline{F}^n$ be a diffeomorphism. We call ϕ a projective mapping if there exists a positive homogeneous scalar function P(x, y) of degree one satisfying $\overline{G}^i = G^i + Py^i$. In this case, P is called the projective factor. Under a projective transformation with projective factor P, the Riemannian curvature tensor of Berwald connection changes as follows by $\overline{K}^i{}_{hjk} = K^i{}_{hjk} + y^i \dot{\partial}_h Q_{jk} + \delta^i_h Q_{jk} + \delta^i_j \dot{\partial}_h Q_k - \delta^i_k \dot{\partial}_h Q_j$, where $Q_i = d_i P - PP_i$ and $Q_{ij} = \dot{\partial}_i Q_j - \dot{\partial}_j Q_i$. A projective transformation with projective factor P is said to be *C*-projective if $Q_{ij} = 0$.

Let *X* be a projective vector field on a Finsler manifold (M, F). Let the vector field *X* in a local coordinate (x^i) on *M* be written in the form $X = X^i(x)\partial_i$. Then the complete lift of *X* is denoted by \hat{X} and locally defined by $\hat{X} = X^i\partial_i + y^j\partial_j X^i\partial_i$. Suppose that $\pounds_{\hat{X}}$ stands for Lie derivative with respect to the complete lift of *X*. Then we have

$$\pounds_{\hat{X}} G^{i} = P y^{i}, \qquad \pounds_{\hat{X}} G^{i}_{\ k} = \delta^{i}_{k} P + y^{i} P_{k}, \qquad \pounds_{\hat{X}} G^{i}_{\ jk} = \delta^{i}_{j} P_{k} + \delta^{i}_{k} P_{j} + y^{i} P_{jk},$$

$$\pounds_{\hat{X}} G^{i}_{\ jkl} = \delta^{i}_{j} P_{kl} + \delta^{i}_{k} P_{jl} + \delta^{i}_{l} P_{kj} + y^{i} P_{jkl},$$

$$(1)$$

$$\pounds_{\hat{X}} K^{i}_{jkl} = \delta^{i}_{j} (P_{l|k} - P_{k|l}) + \delta^{i}_{l} P_{j|k} - \delta^{i}_{k} P_{j|l} + y^{i} \dot{\partial}_{j} (P_{l|k} - P_{k|l}).$$
⁽²⁾

Since $Q_{ij} = P_{i|j} - P_{j|i}$, we have

$$\pounds_{\hat{X}}K^{i}{}_{jkl} = \delta^{i}_{j}Q_{lk} + \delta^{i}_{l}P_{j|k} - \delta^{i}_{k}P_{j|l} + y^{i}\dot{\partial}_{j}Q_{lk}.$$
(3)

We have $\dot{\partial}_j P_{k|l} = P_{jk|l} - P_r G^r_{jkl}$. Contracting *i* and *k* in (3), yields $\pounds_{\hat{X}} K_{jl} = P_{l|j} - nP_{j|l} + P_{jl|s} y^s$. Consequently $\pounds_{\hat{X}} (y^r \dot{\partial}_l K_{jr}) = -(n+1)P_{jl|s} y^s$. Hence $P_{jl|s} y^s = -\frac{1}{n+1} L(\hat{X})(y^r \dot{\partial}_l K_{jr})$ and

$$\pounds_{\hat{X}}\left(K_{jl} + \frac{1}{n+1}y^r \dot{\partial}_l K_{jr}\right) = P_{l|j} - nP_{j|l}, \qquad \pounds_{\hat{X}}\left(K_{lj} + \frac{1}{n+1}y^r \dot{\partial}_j K_{lr}\right) = P_{j|l} - nP_{l|j}.$$
(4)

Using (4) one can obtain

$$P_{j|l} = \frac{1}{1 - n^2} \pounds_{\hat{X}} \left\{ K_{lj} + \frac{1}{n+1} y^r \dot{\partial}_j K_{lr} + nK_{jl} + \frac{n}{n+1} y^r \dot{\partial}_l K_{jr} \right\}.$$
(5)

If $Q_{ij} = 0$, then (3) reduces to $\pounds_{\hat{X}} K^i_{jkl} = \delta^i_l P_{j|k} - \delta^i_k P_{j|l}$. Eliminating $P_{j|l}$ from this equation and using (5), we are led to the following tensor:

$$\widetilde{W}^{i}_{jkl} := K^{i}_{jkl} - \frac{\delta^{i}_{l}}{1 - n^{2}} \left\{ \widetilde{K}_{jk} + \frac{n}{n+1} y^{r} (\dot{\partial}_{k} K_{jr} - \dot{\partial}_{j} K_{kr}) \right\} + \frac{\delta^{i}_{k}}{1 - n^{2}} \left\{ \widetilde{K}_{jl} + \frac{n}{n+1} y^{r} (\dot{\partial}_{l} K_{jr} - \dot{\partial}_{j} K_{lr}) \right\}$$
(6)

where $\tilde{K}_{jk} := nK_{jk} + K_{kj} + y^r \dot{\partial}_j K_{kr}$. Since $y^j y^r \dot{\partial}_k K_{jr} = 0$, if we put $\widetilde{W}^i{}_k := \widetilde{W}^i{}_{jkl} y^j y^l$, then we have

$$\widetilde{W}^{i}_{k} := K^{i}_{k} - \frac{1}{1 - n^{2}} \{ y^{i} \widetilde{K}_{0k} - \delta^{i}_{k} \widetilde{K}_{00} \}.$$
⁽⁷⁾

For $y \in T_x M_0$, the *C*-projective Weyl curvature $\widetilde{W}_y : T_x M \to T_x M$ is defined by $\widetilde{W}_y(u^i \partial_i) = \widetilde{W}_k^i(y)u^k \partial_i$. According to the way we construct \widetilde{W} , it is easy to see that \widetilde{W} is C-projective invariant tensor. A Finsler metric *F* is called *C*-projective Weyl metric if its C-projective Weyl-curvature vanishes.

3. Proof of Theorem 1

Lemma 2. Let F be a C-projective Weyl metric. Then F is a Weyl metric.

Proof. By assumption, we have $K^i{}_k - \frac{1}{1-n^2} \{ y^i \tilde{K}_{0k} - \delta^i_k \tilde{K}_{00} \} = 0$, which contracting it with y_i implies that $F^2 \tilde{K}_{0k} - y_k \tilde{K}_{00} = 0$. Hence $\tilde{K}_{0k} = F^{-2} y_k \tilde{K}_{00}$. Plugging this equation into the first equation, we get $(1 - n^2)K^i{}_k = \tilde{K}_{00}h^i_k$, which means that F is a Weyl metric. \Box

Lemma 3. Let F be a Weyl metric of flag curvature λ . Then C-projective Weyl curvature is given by $\widetilde{W}^i{}_k = \frac{1}{3}F^2y^i\lambda_k$, where $\lambda_k := \dot{\partial}_k\lambda$.

Proof. By assumption, the Riemannian curvature of Berwald connection is in the following form:

$$K^{i}{}_{jkl} = \lambda \left(\delta^{i}_{k} g_{jl} - \delta^{i}_{l} g_{jk} \right) + \lambda_{j} F \left(\delta^{i}_{k} F_{l} - \delta^{i}{}_{l} F_{k} \right) + \frac{1}{3} F^{2} \left(h^{i}{}_{k} \lambda_{jl} - h^{i}{}_{l} \lambda_{jk} \right) + \frac{1}{3} \lambda_{l} F \left(2\delta^{i}{}_{k} F_{j} - 2\delta^{i}{}_{j} F_{k} - g_{jk} \ell^{i} \right) - \frac{1}{3} F \lambda_{k} \left(2\delta^{i}{}_{l} F_{j} - 2\delta^{i}{}_{j} F_{l} - g_{jl} \ell^{i} \right)$$
(8)

where $\lambda_{ij} = \dot{\partial}_j \lambda_i$. Hence, we have

$$K_{k}^{i} = \lambda F^{2}h_{k}^{i},$$

$$K_{jl} = (n-1)(\lambda g_{jl} + FF_{l}\lambda_{j}) + \frac{n-2}{3}(F^{2}\lambda_{jl} + 2FF_{j}\lambda_{l}), \quad K_{00} = \lambda(n-1)F^{2}, \quad \tilde{K}_{00} = \lambda(n^{2}-1)F^{2},$$

$$K_{k0} = \lambda(n-1)FF_{k} + \frac{2n-1}{3}F^{2}\lambda_{k}, \quad K_{0k} = \lambda(n-1)FF_{k} + \frac{n-2}{3}F^{2}\lambda_{k},$$

$$\tilde{K}_{0k} = (n^{2}-1)\left(\lambda FF_{k} + \frac{1}{3}F^{2}\lambda_{k}\right).$$
(10)

Plugging (9) and (10) into (7), we get the result. \Box

Lemma 4. Let (M, F) be a C-projective Weyl manifold with dimension $n \ge 3$. Then F is of constant flag curvature.

Proof. By Lemma 2 and Lemma 3, we have $\widetilde{W}_k^i = \frac{1}{3}F^2y^i\lambda_k$. From assumption, we get $\lambda_k = 0$. It means that F is of isotropic flag curvature. The result follows by Schur's Lemma.

Now, let us consider the case F being of constant flag curvature.

Lemma 5. If *F* is of constant flag curvature $\mathbf{K} = \lambda$, then it is a *C*-projective Weyl metric.

Proof. If *F* is of constant flag curvature λ , then (8) reduces to equation $K^{i}_{jkl} = \lambda(g_{jl}\delta^{i}_{k} - g_{jk}\delta^{i}_{l})$. Hence $K_{jl} = \lambda(1 - n)g_{jl}$, $\tilde{K}_{jk} = \lambda(1 - n^{2})g_{jl}$. Plugging this equation into (6) yields $\widetilde{W}^{i}_{ikl} = 0$. \Box

Proof of Theorem 1. By Lemma 4, Lemma 5 and Akbar-Zadeh's Theorem, proof is complete. \Box

References

- [1] H. Akbar-Zadeh, Sur les espaces de Finsler à courbures sectionnelles constantes, Acad. Roy. Belg. Bull. Cl. Sci. 74 (1988) 271-322.
- [2] B. Najafi, Z. Shen, A. Tayebi, Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties, Geom. Dedicata 131 (2008) 87-97.
- [3] X. Mo, An Introduction to Finsler Geometry, World Scientific Publishers, 2006.
- [4] X. Mo, On the non-Riemannian quantity H of a Finsler metric, Diff. Geom. Appl. 27 (2009) 7-14.