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# Regularity theory for pluriclosed flow

# Théorie de la régularité pour un flot multifermé

## Jeffrey Streets, Gang Tian

Fine Hall, Princeton University, Princeton, NJ 08544, United States

## ARTICLE INFO

Article history: Received and accepted 16 November 2010 Available online 26 November 2010

Presented by Jean-Michel Bismut

#### ABSTRACT

In Streets and Tian (2010) [6] the authors introduced a parabolic flow of pluriclosed metrics. New advancements in the study of this flow are given, including improved regularity results, a gradient property and expanding entropy functional, and a conjectural picture of optimal existence results and their topological consequences. Finally we introduce a family of geometric evolutions in almost Hermitian geometry which provides a general framework for this flow.

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## RÉSUMÉ

Dans Streets et Tian (2010) [6] les auteurs ont introduit un flot parabolique de métriques multifermées. On donne dans cette Note de nouveaux résultats incluant des propriétés de régularité, une propriété de gradient et une fonctionnelle d'entropie croissante, puis une conjecture pour des résultats d'existence et leurs conséquences topologiques. On introduit aussi une famille d'évolutions géométriques dans une géométrie presque hermitienne qui fournit un cadre général à l'étude de ce flot.

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The construction of canonical geometries has had a profound impact on our understanding of complex manifolds. The construction of Kähler–Einstein metrics, constant scalar curvature Kähler metrics, Hermitian Yang–Mills connections, and the use of the Kobayashi–Hitchin correspondence have played a major role in understanding complex manifolds, especially complex surfaces. These strategies share many common threads. In particular, the equations tend to come from mathematical physics, and the existence of solutions is related to certain complex subobjects, i.e. vector bundles or submanifolds. In prior work [6] the authors introduced a certain parabolic evolution equation associated to non-Kähler, Hermitian metrics. In this Note we announce some new results concerning this flow and its possible generalizations, and moreover show that it shares the common threads of the equations above. Specifically we show that our flow is, after gauge transformation, equivalent to the renormalization group flow of a certain nonlinear sigma model which arises in string theory. We also provide a conjectural framework for the existence properties of our flow, and show the relationship of this conjecture to understanding curves on complex surfaces, which has strong consequences for understanding the topology of non-Kähler surfaces.

Let  $(M^{2n}, J)$  be a complex manifold, and let  $\omega$  denote a Hermitian metric on M. The metric  $\omega$  is *pluriclosed* if

 $\partial \overline{\partial} \omega = 0.$ 

E-mail addresses: jstreets@math.princeton.edu (J. Streets), tian@math.princeton.edu (G. Tian).

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This condition is a natural weakening of the Kähler condition, and pluriclosed Hermitian metrics are also called strong Kähler with torsion. Consider the initial value problem

$$\frac{\partial}{\partial t}\omega = \partial\partial^*\omega + \overline{\partial}\overline{\partial}^*\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g, \qquad \omega(0) = \omega_0.$$
(1)

This equation was introduced in [6]. It preserves the pluriclosed condition, and moreover if  $\omega_0$  is Kähler simply reduces to the Kähler–Ricci flow. It is possible to express Eq. (1) using the curvature of the Chern connection. Specifically, let *T* and  $\Omega$  denote the torsion and curvature of the Chern connection, respectively. Let

$$S_{i\bar{j}} = g^{k\bar{l}} \Omega_{k\bar{l}i\bar{j}}, \qquad Q_{i\bar{j}} = g^{k\bar{l}} g^{m\bar{n}} T_{ik\bar{n}} T_{\bar{j}\bar{l}m}.$$

Then a solution to (1) can be expressed

$$\frac{\partial}{\partial t}g = -S + Q.$$

This viewpoint was used in [7] and [6] to prove certain basic regularity theorems for (1), and indeed a wider class of flows of Hermitian metrics. Observe here that *S* is the curvature term appearing in Hermitian Yang–Mills theory on the tangent bundle, the only important difference being that we are not taking the trace with respect to a fixed background metric, but rather the given Hermitian metric. This makes *S* as we have defined it a more intrinsic version of the Hermitian Yang–Mills curvature on the tangent bundle. Therefore Eq. (1) can be thought of as finding solutions to an intrinsic version of Hermitian Yang–Mills theory. This analogy is even stronger on complex surfaces, where  $Q = \frac{1}{2}|T|^2g$ , and hence a metric satisfying  $S - Q = \lambda \omega$  in fact satisfies  $S = f \omega$  for some function *f*. This condition is sometimes called weakly Hermitian Yang–Mills.

As interesting as this viewpoint is, the starting point for our new results is a new interpretation of Eq. (1) using the Bismut connection. This is a crucial observation which clarifies many aspects of (1). Let us recall the Bismut connection. Let  $(M^{2n}, \omega, J)$  be a complex manifold with pluriclosed metric, and let D denote the Levi-Civita connection. The Bismut connection  $\nabla$  is defined by

$$\langle \nabla_X Y, Z \rangle = \langle D_X Y, Z \rangle + \frac{1}{2} d^c \omega (X, Y, Z)$$

where  $d^c \omega(X, Y, Z) := d\omega(JX, JY, JZ)$ . This connection is defined as the unique connection compatible with both *g* and *J* such that the torsion is a three-form, which is moreover closed if  $\omega$  is pluriclosed. Let  $\Omega$  denote the curvature of this connection, and let *P* denote the curvature of the induced connection on the canonical bundle. Specifically, if roman letters *i*, *j*, *k*, etc. indicate vectors in *TM* and Greek letters  $\alpha$ ,  $\beta$ ,  $\gamma$ , etc. indicate vectors in  $T^{1,0}M$ , then

$$P_{ab} = \Omega^{\overline{\alpha}}_{ab\alpha}.$$

By direct calculation one can show that a solution to (1) may be expressed as

$$\frac{\partial}{\partial t}\omega = -P^{1,1} \tag{2}$$

where  $P^{1,1}$  denotes the projection of P onto (1, 1)-forms. This already suggests that the Bismut connection is the appropriate tool for understanding solutions to (1). Another consequence of this new formulation is to enable us to relate (1), or equivalently (2), to a geometric evolution equation naturally associated to a connection with torsion given by a closed three-form, namely the B-field renormalization group flow of string theory. Thus Perelman-type monotone quantities which had previously been discovered for the B-field flow, specifically an energy functional [4] and expanding entropy functional [5], apply to solutions to (1).

To describe the energy functional, first generalize the notation slightly and let  $(M^n, g)$  be a Riemannian manifold, and let T denote a closed three-form on M. Let

$$\mathcal{F}(g, T, f) = \int_{M} \left[ R - \frac{1}{12} |T|^2 + |\nabla f|^2 \right] e^{-f} \, \mathrm{d}V.$$

Furthermore set

$$\lambda(g,T) = \inf_{\{f \mid \int_M e^{-f} dV = 1\}} \mathcal{F}(g,T,f).$$

As shown in [4], the gradient flow of  $\lambda$  is the *B*-field renormalization group flow, which is the coupled system of evolution equations

$$\frac{\partial}{\partial t}g_{ij} = -2\operatorname{Rc}_{ij} + \frac{1}{2}H_i^{pq}H_{jpq}, \qquad \frac{\partial}{\partial t}T = \Delta_{LB}T$$

where  $\Delta_{LB}$  denotes the Laplace–Beltrami operator of the time-dependent metric.

$$\mathcal{M} := \frac{\{(g, T) \mid g \in \text{Met}, T \in \Lambda^3, dT = 0\}}{\text{Diff}_+(M)}$$

where  $\text{Diff}_+$  is the group of oriented diffeomorphisms of M, acting naturally on g and T. The pair (g(t), T(t)) is a solution of the gradient flow of  $\lambda$  acting on  $\mathcal{M}$ .

Furthermore, in [2] Feldman, Ilmanen and Ni gave a generalization of Perelman's steady and shrinking entropies to an entropy modelled on expanding solitons. Surprisingly, this expanding entropy has an extension to the B-field renormalization group flow [5], which hence applies to solutions of (1). These results are an important step in understanding the singularity formation of solutions to (1). In particular they automatically imply that any steady or expanding breather solution is automatically a gradient soliton, and furthermore imply strong restrictions on long-time solutions.

We move now to the question of long-time existence of (1). Recall that theory of Kähler–Ricci flow is considerably more developed than the general study of Ricci flow, which is largely due to the reduction of the Kähler–Ricci flow to a scalar equation. Inspired by this, we will introduce a certain potential function  $\phi$  along a solution to (1) and prove a regularity theorem in term of this potential and the torsion. Let  $(M^{2n}, \tilde{\omega}, J)$  be a complex manifold with pluriclosed metric, and let  $\omega(t)$  be a solution to (1). We define

$$\frac{\partial}{\partial t}\phi - \Delta\phi = \operatorname{tr}_{\omega}\widetilde{\omega} - n, \qquad \phi(0) = 0. \tag{3}$$

Here  $\Delta$  is the canonical Laplacian associated to the time dependent metric  $\omega(t)$ , i.e.  $\Delta = tr_{\omega} \partial \overline{\partial}$ .

**Theorem 2.** (See [8, Theorem 1.3].) Let  $(M^{2n}, \tilde{g}, J)$  be a compact complex manifold and suppose g(t) is a solution to (1) on  $[0, \tau)$  and suppose there is a constant C such that

$$\sup_{M\times[0,\tau)} |\phi| \leqslant C, \qquad \sup_{M\times[0,\tau)} |T|^2 \leqslant C.$$

Then  $g(t) \rightarrow g(\tau)$  in  $C^{\infty}$ , and the flow extends smoothly past time  $\tau$ .

The proof is a delicate application of the maximum principle. Let  $\tilde{\omega}$  denote some background Hermitian metric. We first derive a differential inequality for  $\operatorname{tr}_{\tilde{\omega}}\omega$  based on Laplacian estimates for the Kähler–Ricci flow. Next we show a differential inequality for log  $\frac{\det g}{\det g}$  which then implies  $L^{\infty}$  estimates for the time-dependent metric. By adapting the Calabi  $C^3$  estimates for the complex Monge–Ampere equation to our setting we are able to derive a  $C^1$  estimate at this point, after which we can apply Schauder estimates to obtain the full regularity. It is not unreasonable to expect a priori estimates for the torsion along solutions to (1), and in the presence of such an estimate our theorem says that the long-time existence behavior is governed by one function.

Ideally one would like to understand what the optimal existence and regularity theorems are for (1). To formulate a precise conjecture we look to the study of Kähler–Ricci flow. Suppose  $(M^{2n}, \omega_0, J)$  is a Kähler manifold. Recall the Kähler–Ricci flow equation

$$\frac{\partial}{\partial t}\omega = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\det g, \qquad \omega(0) = \omega_0.$$
(4)

Associated to any solution of (4) is an ODE in  $H^2(M, \mathbb{R})$  which has solution

$$\left[\omega(t)\right] = \left[\omega_0\right] - tc_1(M).$$

The optimal regularity theorems for Kähler–Ricci flow assert that as long as the solution to this ODE remains in the Kähler cone, the solution exists up to that time ([11], see also [10]).

To draw the analogy with (1), first note that a pluriclosed metric defines a class in the finite dimensional Aeppli cohomology group

$$\mathcal{H}_{\partial+\bar{\partial}}^{1,1} = \frac{\{\operatorname{Ker} \partial \bar{\partial} : \Lambda_{\mathbb{R}}^{1,1} \to \Lambda_{\mathbb{R}}^{2,2}\}}{\{\partial \alpha + \bar{\partial} \bar{\alpha} \mid \alpha \in \Lambda^{0,1}\}}$$

Define the space  $\mathcal{P}_{\partial+\bar{\partial}}$  to be the cone of the classes in  $\mathcal{H}_{\partial+\bar{\partial}}$  which contain positive definite elements. As in the Kähler case, one can associate an ODE in  $\mathcal{H}^{1,1}_{\partial+\bar{\partial}}$  to solution of (1), and it is natural to conjecture that the maximal existence time is characterized by the first time at which the boundary of  $\mathcal{P}_{\partial+\bar{\partial}}$  is reached.

**Conjecture 3.** Let  $(M^{2n}, g_0, J)$  be a compact complex manifold with pluriclosed metric. Let

$$\tau^* := \sup_{t \ge 0} \{ t \mid [\omega_0 - tc_1] \in \mathcal{P}_{\partial + \overline{\partial}} \}.$$

Then the solution to (1) with initial condition  $g_0$  exists on  $[0, \tau^*)$ , and  $\tau^*$  is the maximal time of existence.

This conjecture is strongly motivated by the close connection between (1) and Kähler–Ricci flow. Indeed, the conjecture holds for solutions to Kähler–Ricci flow ([11], see also [10]).

A positive resolution of Conjecture 3 would have topological consequences for non-Kähler surfaces. This is because, to complete the analogy with other equations in complex geometry, we can characterize the cone  $\mathcal{P}_{\partial+\bar{\partial}}$  in terms of complex subobjects on complex surfaces. In the theorem below  $\gamma_0$  is a positive generator of

$$\Gamma = \frac{d\{\Lambda_{\mathbb{R}}^1\} \cap \Lambda_{\mathbb{R}}^{1,1}}{i\partial \overline{\partial} \Lambda_{\mathbb{R}}^0}$$

which is naturally isomorphic to  $\mathbb{R}$  on a non-Kähler surface.

**Theorem 4.** (See [8, Theorem 5.6].) Let  $(M^4, J)$  be a complex non-Kähler surface. Suppose  $\phi \in \Lambda^{1,1}$  is pluriclosed. Then  $\phi \in \mathcal{P}_{\partial + \overline{\partial}}$  if and only if

- $\int_M \phi \wedge \gamma_0 > 0;$
- $\int_{D} \phi > 0$  for every effective divisor with negative self intersection.

Using this characterization, we can show that Conjecture 3 implies that a Class VII<sup>+</sup> surface without curves admits longtime nonsingular solutions to (1). By exploiting certain monotone quantities and the Perelman quantities noted above, it follows that Class VII<sup>+</sup> surfaces admit no such solutions. We conclude that a positive resolution of Conjecture 3 implies the existence of a curve on a Class VII<sup>+</sup> surface. Due to the results of Nakamura [3], and Dloussky, Oeljeklaus, and Toma [1], the classification of these surfaces is reduced to finding sufficiently many curves. In particular one can show that Conjecture 3 implies the classification of Class VII<sup>+</sup> surfaces with  $b_2 = 1$ , a result recently obtained by Teleman using gauge theory [9]. Indeed it implies more, specifically the existence of a curve on *any* Class VII<sup>+</sup> surface. One expects that a sufficiently clear picture of the singularity formation of (1) in fact implies the existence of enough curves to imply the full classification of these surfaces.

We close by announcing a wider framework in which to view Eq. (1), which is a natural flow of almost Hermitian pairs (g, J). Let  $(M^{2n}, g, J)$  be an almost Hermitian manifold. Let  $\nabla$  denote the Chern connection associated to (g, J), which is the unique connection compatible with both g and J such that the torsion satisfies  $T^{1,1} \equiv 0$ . Let  $\Omega$  denote the (4, 0)-curvature tensor associated to this connection, and let  $S = \operatorname{tr}_{\omega} \Omega$ . Furthermore, let Q denote any quadratic expression in the torsion T of  $\nabla$ , and let  $\mathcal{P} = \operatorname{tr}_{\omega} \nabla N$ , where N denotes the Nijenhuis tensor associated to J. Consider the initial value problem

$$\frac{\partial}{\partial t}g = -S + Q + F, \qquad \frac{\partial}{\partial t}J = -\mathcal{P}, \qquad g(0) = g_0, \qquad J(0) = J_0.$$
(5)

This is a degenerate parabolic system of equations for (g, J), with degeneracy arising from the action of the diffeomorphism group. One can show general short-time existence for (5) as well as smoothing estimates, generalizing Theorem 1.1 of [7]. Of course if the initial complex structure is integrable, then the resulting evolution fixes J, and one is reduced to a flow of Hermitian metrics. For a specific choice of Q, Eq. (1) arises. An interesting further direction to pursue is to understand what other integrability conditions on  $\omega$  and J are preserved by solutions to (5). There is a different instance of Eq. (5) which preserves the almost Kähler condition, thus naturally extending Kähler Ricci flow onto symplectic manifolds. More detailed discussions on (5) will appear elsewhere.

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