Homological Algebra/Topology

# Lambda algebra and the Singer transfer ${ }^{\text {H/ }}$ 

## Lambda algèbre et le transfert de Singer

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## A R T I C L E I N F O

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#### Abstract

We modify Singer's idea to give a direct description of the lambda algebra using modular invariant theory. As an application, we describe the algebraic transfer in purely invarianttheoretic framework, thus, provides an effective computational tool for the algebraic transfer. The induced action of the Steenrod algebra on lambda algebra is also investigated and clarified.


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## R É S U M É

Utilisant la théorie d'invariants modulaires, nous modifions l'idée de Singer pour donner une description directe de la lambda algèbre. En application, nous décrivons les transferts algébraiques à l'aide de la théorie d'invariants, et ainsi fournir une méthode efficace pour les calculer. L'action induite de l'algèbre de Steenrod sur la lambda algèbre est également étudiée.
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## 1. Introduction

Let $\Lambda_{+}$be the graded tensor algebra over $\mathbb{F}_{2}$ on symbols $\lambda_{i}$ of degree $i, i \geqslant-1$, modulo the two-sided ideal generated by $\lambda_{s} \lambda_{t}-\sum_{j}\binom{j-t-1}{2 j-s} \lambda_{s+t-j} \lambda_{j}$, for any $s, t \geqslant-1$. Here $\binom{n}{k}$ is interpreted as the coefficient of $x^{k}$ in the expansion of $(1+x)^{n}$ so that it is well-defined for all integers $n$ and all non-negative integers $k$. The lambda algebra of Bousfield et al. [2] is the quotient of $\Lambda_{+}$by the right ideal generated by $\lambda_{-1}$ [3]. Let $\Lambda_{s}$ denote the vector space spanned by all monomials in $\lambda_{i}$ of length $s$. It is well known that $\Lambda_{s}$ has a basis consisting of all admissible monomials, i.e. those of the form $\lambda_{i_{1}} \cdots \lambda_{i_{s}}$, where $i_{j} \leqslant 2 i_{j+1}$ for all $1 \leqslant j<s-1$. It should be noted that our definition of lambda algebra follows that of Singer [13], which is opposite (by the canonical reversing-order map) to the original version in [2].

Let $M$ be a graded, connected right module over the $\bmod 2$ Steenrod algebra $\mathcal{A}$. Then we can define a differential $\delta_{s}: \Lambda_{s} \otimes M \rightarrow \Lambda_{s+1} \otimes M$ by claiming that it is a $\Lambda$-map, and that $\delta(1 \otimes x)=\sum_{i \geqslant-1} \lambda_{i} \otimes x S q^{i+1}$. When $M=\mathbb{F}_{2}, \delta$ is just the map induced by the multiplication by $\lambda_{-1}$ in $\Lambda_{+}$. There is a natural isomorphism $\operatorname{Ext}_{\mathcal{A}}^{s, t}\left(\mathbb{F}_{2}, M\right) \rightarrow H^{s}\left(\Lambda_{*} \otimes M\right)$. In particular, when $M=H_{*}(X)$, where $X$ is a (2-completed) spectrum, we obtain a chain complex whose homology is the $E^{2}$ page $\mathrm{Ext}_{\mathcal{A}}^{s, t}\left(\mathbb{F}_{2}, H_{*}(X)\right)$ of the Adams spectral sequence abutting to $\pi_{*}^{s}(X)$.

[^0]Because of naturality, the stable transfer $B(\mathbb{Z} / 2)_{+}^{s} \rightarrow S^{0}$ must induce a map between the $E^{2}$ terms of the corresponding Adams spectral sequences, which in turn should be induced by a certain map $\Lambda_{*} \otimes H_{*}\left(B(\mathbb{Z} / 2)^{s}\right) \rightarrow \Lambda_{*+s}$. The purpose of this paper is to construct such a map $\varphi_{s}: H_{*}\left(B(\mathbb{Z} / 2)^{s}\right) \rightarrow \Lambda_{s}$ which can be considered as the $E^{1}$ level of Singer's algebraic transfer.

The image of the Singer transfer in small ranks have been investigated extensively, see, for example, $[14,6,1,4,7,5,12,11,8]$. We will give several examples to show the effectiveness of our approach in higher ranks.

## 2. An alternate construction of the lambda algebra

In [13], Singer has already given an invariant-theoretic description of the lambda algebra, but his construction is not quite easily applicable to the transfer. We give here another construction of the lambda algebra. This construction is probably well known to the experts but we have not been able to find a written account. Using our description, we are able to explain the relationship between Wellington's formal Steenrod action [16] and Singer's Steenrod operation on lambda algebra [13,18].

Let $H^{*}\left(B(\mathbb{Z} / 2)^{s}\right)=P_{s}=\mathbb{F}_{2}\left[x_{s}, \ldots, x_{1}\right]$ be the polynomial ring on $s$ generators $x_{1}, \ldots, x_{s}$, where each $x_{i}$ has degree 1 . It is well known that $P_{s}$ has the structure of an $\mathcal{A}\left[G L_{s}\right]$-algebra, where $G L_{s}$ denotes the usual general linear group over $\mathbb{F}_{2}$. Let $S(s)$ be the multiplicative subset of $P_{s}$ generated by all non-zero linear forms in $P_{s}$. Then $\Phi_{s}:=P_{s}\left[S(s)^{-1}\right]$ is again an $\mathcal{A}\left[G L_{s}\right]$-algebra (Wilkerson [17]). Following Singer [13] we have $\Delta_{s}:=\Phi_{s}^{U_{s}} \cong \mathbb{F}_{2}\left[v_{1}^{ \pm 1}, \ldots, v_{s}^{ \pm 1}\right]$, where $v_{k}=V_{k} / V_{1} \cdots V_{k-1}$ and $V_{n}=\prod_{k_{i} \in \mathbb{F}_{2}}\left(k_{1} x_{1}+\cdots+k_{n-1} x_{n-1}+x_{n}\right)$ being the invariants of $P_{s}$ under the action of the group of all upper-triangular matrices $U_{s}$ (see [10]). We can assemble $\Delta_{s}$ together to form an algebra $\Delta=\mathbb{F}_{2}\left[v_{1}^{ \pm 1}, \ldots, v_{s}^{ \pm 1}, \ldots\right]$ with multiplication

$$
\begin{equation*}
v_{1}^{i_{1}} \cdots v_{p}^{i_{p}} \circ v_{1}^{i_{p+1}} \cdots v_{q}^{i_{p+q}} \rightarrow v_{1}^{i_{1}} \cdots v_{p}^{i_{p}} v_{p+1}^{i_{p+1}} \cdots v_{p+q}^{i_{p+q}} . \tag{1}
\end{equation*}
$$

Let $\mathcal{L}_{1}=\Delta_{1}$. For $s \geqslant 2$, let $\mathcal{L}_{s}$ be the quotient of $\Delta_{s}$ by the two-sided ideal generated by $\Phi_{2}^{G L_{2}}$. Our first result relates this construction with the Steinberg idempotent.

Proposition 2.1. There is a natural isomorphism of $\mathcal{A}$-modules $\mathcal{L}_{s} \rightarrow \Phi_{s} S t$, where $\Phi_{s}$ St is the Steinberg summand of $\Phi_{s}$.
Hints of such a relation has been given in [9] and [15]. Consider the $\mathbb{F}_{2}$-linear map $\mathcal{L}_{s} \xrightarrow{f_{s}} \Lambda_{s}$ which sends $v_{1}^{i_{1}} \ldots v_{s}^{i_{s}}$ to $\lambda_{-i_{1}-1} \cdots \lambda_{-i_{s}-1}$, where it is understood that the expression on the right is zero if there is some $i_{k} \geqslant 0$. A sequence $I=\left(i_{1}, i_{2}, \ldots, i_{s}\right)$ that does not satisfy above condition is said to be completely negative. If $i_{k}>2 i_{k+1}$, then $-i_{k}-1 \leqslant$ $2\left(-i_{k+1}-1\right)$. Hence, $f_{s}$ sends admissible monomials $v^{I}$ (i.e. those of the form $v^{I}=v^{i_{1}} \ldots v^{i_{s}}$, where $i_{k}>2 i_{k+1}$ for all $1 \leqslant k \leqslant s-1$ ) to admissible elements $\lambda_{-I-1}$ in $\Lambda_{s}$. Clearly, $f_{s}$ is onto. Let $K_{s}$ be the vector space spanned by all completely negative and admissible monomials $v^{I}$; then we obtain:

Proposition 2.2. $K_{s}$ is a quotient $\mathcal{A}$-module of $\Delta_{s}$. The restriction of $f_{s}$ on $K_{s}$ induces an isomorphism between $K_{s}$ and the lambda algebra.

This proposition also provides $\Lambda_{*}$ with the structure of an $\mathcal{A}$-algebra under the multiplication given in (1). In [13], Singer introduced the action of the Steenrod algebra on the dual of $\Lambda_{*}$ that is linear for the differential (see also [18]). Our next result gives a recursive formula to calculate Steenrod operations on $\Lambda_{*}$. Furthermore, it implies that the action in Proposition 2.3 coincides with (the dual of) Singer's.

Proposition 2.3. For $a, s \geqslant 0$ and any $\lambda$ in $\Lambda_{*}$, the right action of the Steenrod algebra on lambda algebra is determined as follows:

$$
\left(\lambda_{s} \lambda\right) S q^{a}=\sum_{t}\binom{s-a}{a-2 t} \lambda_{s-a+t}\left(\lambda S q^{t}\right)
$$

In [16], using the Nishida relations, Wellington forced a formal action of the Steenrod algebra on the lambda algebra as well as the Dyer-Lashof algebra, and for a long time it is not clear what the relationship between Wellington and Singer's action is (see comment in the last section of [18]). The above analysis shows that the two actions are almost the same, except for the use of the generalized binomial coefficients $\binom{n}{k}$ (which is defined for all non-negative integers $n$ and $k$ ).

## 3. The Singer transfer

In this section, we review the definition of the Singer transfer and construct a map $H_{*}\left(B(\mathbb{Z} / 2)^{s}\right) \rightarrow \Lambda_{S}$ that induces the Singer transfer. Write $H_{s}=H_{*}\left(B(\mathbb{Z} / 2)^{s}\right)=\Gamma\left[e_{s}, \ldots, e_{1}\right]$ - the divided power algebra on $s$ generators, where we use the canonical dual basis. Let $\hat{P}$ be $\mathcal{A}$-module extension of $P_{1}$ by formally adding a generator $x_{1}^{-1}$ in degree -1 and require that $S q\left(x_{1}^{-1}\right) S q\left(x_{1}\right)=1$, and let $\hat{H}$ be the dual of $\hat{P}$. Tensor the short exact sequence $\Sigma^{-1} \mathbb{F}_{2} \rightarrow \hat{H} \rightarrow H_{1}$ with $H_{n-1}$ and then with $\Lambda_{*} \otimes M$, for some $\mathcal{A}$-module $M$, we have a short exact sequence of differential graded modules

$$
\Lambda_{*} \otimes M \otimes H_{n-1} \rightarrow \Lambda_{*} \otimes M \otimes H_{n-1} \otimes \hat{H} \rightarrow \Lambda_{*} \otimes M \otimes H_{n}
$$

Taking homology, one has a connecting homomorphism

$$
\operatorname{Ext}_{\mathcal{A}}^{s-n, t}\left(\mathbb{F}_{2}, M \otimes H_{n}\right) \rightarrow \operatorname{Ext}_{\mathcal{A}}^{s-n+1, t+1}\left(\mathbb{F}_{2}, M \otimes H_{n-1}\right)
$$

Splicing these connecting homomorphisms for $n$ from $s$ to 1 , we obtain a homomorphism

$$
\operatorname{Ext}_{\mathcal{A}}^{0, t}\left(\mathbb{F}_{2}, M \otimes H_{s}\right) \rightarrow \operatorname{Ext}_{\mathcal{A}}^{s, t+s}\left(\mathbb{F}_{2}, M\right)
$$

When $M=\mathbb{F}_{2}$, this is called the Singer transfer, and it is induced by a map $\varphi_{s}: \Gamma\left[e_{s}, \ldots, e_{1}\right] \rightarrow \Lambda_{s}$.
Theorem 3.1. The representation $\varphi_{s}$ for the Singer transfer is given in terms of generating function as follows:

$$
\begin{equation*}
\varphi_{s}: e\left[x_{s}, x_{s-1}, \ldots, x_{1}\right] \rightarrow \lambda\left[v_{1}, v_{2}, \ldots, v_{s}\right] \tag{2}
\end{equation*}
$$

That is, the transfer $\varphi_{s}$ sends an element $z=e^{(I)} \in H_{*}\left(B(\mathbb{Z} / 2)^{s}\right)$ to the sum of all $\lambda_{J} \in \Lambda_{s}$ such that $x^{I}$ is a non-trivial summand in the expansion of $v^{J}$ in the variables $x_{1}, \ldots, x_{s}$. In other words, $\varphi_{s}: z \rightarrow \sum_{J}\left\langle z, v^{J}\right\rangle \lambda_{J}$. This formula is quite suitable for computer calculation.

## 4. Applications

Using the representation of the Singer transfer constructed in Section 3 to study the image of the Singer transfer, we obtain the description of the image of the transfer at some degrees. The following theorem is the main result of this section:

## Theorem 4.1. The elements

(i) $\mathrm{Ph}_{2} \in \operatorname{Ext}_{\mathcal{A}}^{5,16}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)$,
(ii) $h_{1} P h_{1} \in \operatorname{Ext}_{\mathcal{A}}^{6,16}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)$,
(iii) $h_{0} P h_{2} \in \operatorname{Ext}_{\mathcal{A}}^{6,17}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)$, and
(iv) $h_{0}^{2} P h_{2} \in \operatorname{Ext}_{\mathcal{A}}^{7,18}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)$
are not in the image of the algebraic transfer.
Part (i) was the main result of [12], but our method is much less computational. The last three parts are new. They are interesting because the domain of the Singer transfer beyond rank 4 is generally not accessible.

The contents of this Note will be published in detail elsewhere.

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